From Manifolds to Condensed Spaces

Part I of *Toward a Condensed Representation Theory of Physics*

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07 July 2026

Abstract

This is the first paper of a modular, seven-part program built on the governing perspective that *physics is the study of realizations of condensed mathematical structures.* We ask a foundational question: is the smooth manifold the most fundamental language for the arena of quantum gravity? We recall precisely what the smooth-manifold formalism for spacetime presupposes — a fixed underlying point-set, C^∞-compatible local charts, second countability and Hausdorffness — and we isolate three structural tensions with Planck-scale and quantum-informational physics: (i) the C^∞ chart hypothesis is an idealization with no operational content below the Planck length; (ii) the manifold notion of locality clashes with the nonlocal correlation structure of entangled quantum states; and (iii) the category of topological (abelian) groups in which fields naturally live is not abelian, obstructing the homological algebra a quantum theory of geometry would need. We then develop, in a self-contained and rigorous manner, the *condensed sets* of Clausen and Scholze: profinite and extremally disconnected sets, the pro-étale site of a point, the categories Cond(Set) and Cond(Ab), and the comparison functor Top $\to$ Cond(Set), which is faithful in general and fully faithful on compactly generated spaces (Scholze's Proposition 1.7). Our central deliverable is an explicit, theorem-level comparison — with a summary table and a commutative diagram — of the arrow Manifold $\longrightarrow$ Condensed Space, arguing that the condensed world is a strictly larger, better-behaved category into which the *underlying topological spaces* of manifolds embed fully faithfully (so all topological information is preserved), while manifolds themselves embed faithfully (their smooth structure being extra data), and whose extra room is precisely where discreteness, non-separatedness and quantum nonlocality can be accommodated. Throughout we enforce a strict epistemic discipline: every claim is labelled [EST] (established mathematics), [HEU] (heuristic physical reading), or [SPEC] (speculative hypothesis of this program). The condensed mathematics is entirely [EST]; the proposal that Planck-scale spacetime *is* condensed rather than smooth is [SPEC], clearly bounded and forward-referenced to Parts VI–VII. Part I is the foundation on which Parts II through VII compose.

Contents

1 Introduction

1.1 The modular program and its governing perspective

This paper is Part I of a modular seven-part research program, *Toward a Condensed Representation Theory of Physics*. The program is organized around a single governing slogan:

> *Physics is the study of realizations of condensed mathematical structures.*

The word *modular* is load-bearing and deliberate. This is not a single monolithic theory but a hierarchy of seven papers, each of which composes upon its predecessors:

$$\mathrm{I} \to \mathrm{II} \to \mathrm{III} \to \mathrm{IV} \to \mathrm{V} \to \mathrm{VI} \to \mathrm{VII}.$$

Part I (this paper) replaces the smooth manifold as the ambient notion of *space*. Part II reformulates *observables* as condensed sheaves on the resulting site. Part III proposes *information* (entanglement, state reconstruction) as compatibility data in condensed cohomology. Part IV re-expresses *fields* as condensed/solid/liquid modules on a condensed base. Part V asks whether *geometry* can be recovered as a realization (Tannakian-style fiber) functor. Part VI synthesizes these into a candidate representation-theoretic foundation for quantum gravity, and Part VII poses the resulting research program as a capstone. Each arrow above is a genuine composition: the output category of one paper is the input category of the next. Part I therefore has one job — to fix, rigorously and honestly, the *ambient category of spaces* on which everything downstream is built.

1.2 The epistemic discipline: EST, HEU, SPEC

The single most important structural feature of this program is that it keeps *established mathematics* strictly separated from *speculative physics*. We adopt a three-valued status calculus, inherited from the collaboration's prior work, and we attach one of three labels to every substantive claim:

- [EST] — *established*: a citable, rigorous theorem of mathematics (or a rigorously proved physics-adjacent statement). All of the condensed mathematics in this paper is [EST].
- [HEU] — *heuristic*: a physically motivated dictionary entry or analogy, not itself a theorem.
- [SPEC] — *speculative*: an ontological or physical hypothesis of this research program, with no established precedent.

The three warrants are ordered [EST] < [HEU] < [SPEC], and composition is *monotone, worst-component-wins*: no chain of reasoning is stronger than its weakest link. Concretely, the mathematics of condensed sets is beyond reproach ([EST]); the proposal that spacetime *is* a condensed anima ([SPEC]) inherits the weakest status of any chain it participates in. We state this explicitly so that no reader mistakes the rigor of the mathematical substrate for endorsement of the physical hypothesis. This paper is the *first half* of the series in the sense that it rests almost entirely on accepted theory; the genuinely speculative payloads are concentrated in Parts VI–VII, and we forward-reference them wherever a bounded speculation appears.

1.3 The question

Classical general relativity models spacetime as a smooth, connected, four-dimensional Lorentzian manifold (M, g). This has been extraordinarily successful. But a smooth manifold is a very rigid and very particular kind of object, and it makes strong presuppositions — an underlying set of points, a topology, a maximal C^∞ atlas — that are almost never questioned because at accessible scales they are unimpeachable. The thesis of Part I is that these presuppositions are an *idealization* whose operational content dissolves at the Planck scale, and that there already exists, in pure mathematics, a strictly more flexible and better-behaved notion of "space" — the *condensed sets* of Clausen and Scholze [1, 2] — into which manifolds embed faithfully and whose additional structure is exactly suited to the phenomena (discreteness, non-separatedness, nonlocality, homological well-behavedness) that stress the manifold picture.

We emphasize at the outset the correct and honest form of this claim. We do *not* claim that manifolds are "replaced" or "wrong." We claim, and prove (theorems 6.4 and 6.5), that the *underlying topological space* of a manifold embeds *fully faithfully* into the category of condensed sets, so that no topological information is lost, while the category Mfd itself embeds only *faithfully* — the smooth structure being extra data not seen by the topological comparison functor (and recovered, if at all, only in Part V). We then argue — at status [HEU]/[SPEC] — that the strictly larger condensed category has room for structures a manifold cannot express.

1.4 Contributions and outline

The contributions of this paper are:

1. A precise recollection (section 3) of the smooth-manifold formalism for spacetime and an itemized list of what it presupposes.

2. An analysis (section 4) of three structural tensions — the C^∞ idealization at the Planck scale, manifold locality versus quantum nonlocality, and the failure of Top(Ab) to be abelian — each carefully labelled by epistemic status.

3. A self-contained, rigorous development (sections 5 to 7) of profinite and extremally disconnected sets, the pro-étale site of a point, condensed sets and condensed abelian groups, and the sheaf-theoretic "space from local data" mechanism.

4. The central deliverable (sections 8 and 9): an explicit theorem-level comparison Manifold $\to$ Condensed Space, with a table (table 1) and a commutative diagram (fig. 1), together with the main structural results, including the comparison functor (faithful in general, fully faithful on compactly generated spaces) and the abelian-category repair theorem.

5. Accompanying formal Haskell code (section 9.3) realizing profinite sets as inverse limits, condensed-set probes, and a toy sheaf-gluing computation.

Section 10 collects limitations, explicit non-identifications with neighboring programs (cohesive homotopy type theory; topos quantum theory; causal sets; noncommutative geometry), and the status-composition bookkeeping; section 11 concludes and forward-references Parts II–VII.

2 Mathematical Framework

We fix notation and recall the categorical scaffolding — sites, sheaves, descent — on which both the manifold and the condensed pictures rest. This section is entirely [EST].

2.1 Categories, sites and sheaves

Definition 2.1 (Grothendieck topology / site)**.** A *Grothendieck topology* on a category $\mathcal{C}$ is an assignment, to each object $U \in \mathcal{C}$, of a collection of *covering families* $\{U_i \to U\}_{i \in I}$, satisfying:

1. (Identity) $\{\mathrm{id}_U : U \to U\}$ is a covering family.
2. (Stability) if $\{U_i \to U\}$ covers and $V \to U$ is any morphism, then the pullbacks $\{U_i \times_U V \to V\}$ exist and form a covering family of V.
3. (Transitivity) if $\{U_i \to U\}$ covers and each $\{U_{ij} \to U_i\}$ covers, then $\{U_{ij} \to U\}$ covers.

A category equipped with a Grothendieck topology is a *site*.

Definition 2.2 (Sheaf on a site)**.** A presheaf $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ is a *sheaf* if for every covering family $\{U_i \to U\}$ the diagram

$$F(U) \longrightarrow \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \times_U U_j)$$

is an equalizer. Equivalently: any family of local sections $s_i \in F(U_i)$ that agree on overlaps glues to a unique global section $s \in F(U)$. We write $\mathrm{Sh}(\mathcal{C})$ for the category of sheaves of sets on the site $\mathcal{C}$.

The equalizer condition of theorem 2.2 is the categorical form of the slogan "*space is what you can reconstruct from compatible local data.*" The founding reference is SGA4 [8]; for modern treatments see Vistoli [9], Mac Lane–Moerdijk [10] and the Stacks Project [11]. A basic and important fact [EST] is that a topos of sheaves depends only on the Grothendieck topology *generated*, not on the particular generating pretopology; this lets us present the condensed topos by several equivalent sites in section 6.

Remark 2.3 (Descent as the organizing principle). The passage from a presheaf to a sheaf — *sheafification* — and the equalizer condition together encode *descent*: global objects are determined by, and glued from, local objects plus compatibility on overlaps. Every notion of "space" considered in this paper (topological space, manifold, condensed set) is, at bottom, a prescription for which local data are allowed and how they are required to glue. The thesis of this program is that the right such prescription for physics is the condensed one.

2.2 Notational conventions

We write Fin for finite sets, ProFin for profinite sets, ExtDisc for extremally disconnected profinite sets, CompHaus for compact Hausdorff spaces, Top for the category of *all* topological spaces with continuous maps, CGWH $\subseteq$ Top for the full subcategory of compactly generated weak Hausdorff spaces, and Mfd for smooth manifolds with smooth maps. The categories of condensed sets and condensed abelian groups are Cond(Set) and Cond(Ab). For a topological space X and a profinite set S we write Cont(S, X) for the set of continuous maps $S \to X$. All rings are commutative unless stated otherwise. We use $\underline{X}$ for the condensed set associated to a topological space X (theorem 6.3).

3 Classical Spacetime as a Differentiable Manifold

We recall the smooth-manifold formalism, in the precise form used in general relativity, and we isolate exactly what it presupposes. All statements in this section are [EST] (standard differential geometry) except where a physical reading is flagged [HEU].

Definition 3.1 (Smooth manifold)**.** A *smooth n-manifold* is a topological space M that is

1. *Hausdorff* (distinct points have disjoint neighborhoods);
2. *second countable* (the topology has a countable base);
3. *locally Euclidean*: every point has an open neighborhood homeomorphic to an open subset of $\mathbb{R}^n$,

together with a maximal C^∞ *atlas* $\{(U_\alpha, \varphi_\alpha)\}$, i.e. a covering by charts $\varphi_\alpha : U_\alpha \xrightarrow{\sim} \varphi_\alpha(U_\alpha) \subseteq \mathbb{R}^n$ whose transition maps $\varphi_\beta \circ \varphi_\alpha^{-1}$ are C^∞ diffeomorphisms on overlaps.

Definition 3.2 (Lorentzian spacetime)**.** A *spacetime* is a connected smooth 4-manifold M equipped with a smooth Lorentzian metric g, a nondegenerate symmetric $(0, 2)$-tensor field of signature $(-, +, +, +)$, together with a time orientation. The Einstein field equations

$$G_{\mu\nu} + \Lambda g_{\mu\nu} \; = \; 8\pi \, T_{\mu\nu}$$

relate the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R \, g_{\mu\nu}$ to the stress–energy $T_{\mu\nu}$. Well-posedness of the initial value problem (Choquet-Bruhat) requires M to be globally hyperbolic, hence diffeomorphic to $\Sigma \times \mathbb{R}$ for a Cauchy hypersurface Σ.

Remark 3.3 (What the manifold formalism presupposes). Theorem 3.1 bundles together several strong, logically independent hypotheses that are worth naming individually, because the condensed reformulation will relax them independently:

(P1) **A completed point-set.** M has a fixed underlying set of points, of cardinality that of the continuum, given *prior* to any physical process. Points exist whether or not anything localizes there.

(P2) **Local Euclidean charts.** Space looks like $\mathbb{R}^n$ arbitrarily far into the small; there is a well-defined tangent space, and hence a well-defined notion of "infinitesimally near" at every scale.

(P3) C^∞ **transition data.** Overlaps are related by infinitely differentiable maps; the smooth structure is part of the datum, not derived.

(P4) **Separation and countability.** Hausdorffness forbids "infinitely close but distinct" points; second countability forbids "too many" points.

(P5) **Manifold locality.** Physical influence propagates through open neighborhoods; the metric's light cones bound it. Spacelike-separated regions are causally independent.

Each of (P1)–(P5) is empirically superb at accessible scales. Each is, we will argue, an *idealization* without operational content at the Planck scale, and each is relaxed, in a controlled way, by the condensed picture.

Example 3.4 (The tangent bundle as an infinitesimal idealization)**.** The tangent space T_pM is defined via germs of smooth curves, or equivalently derivations of the algebra $C_p^\infty(M)$ of germs of smooth functions at p. Both definitions invoke the full C^∞ structure at arbitrarily small scales (P2, P3). Operationally, however, T_pM is only ever probed through finite-resolution measurements; the claim that a genuine $\mathbb{R}^n$ of tangent directions exists "at" p is a mathematical convenience, not a measured fact. [HEU]

4 The Limits of Manifolds

We now articulate three structural tensions between the manifold formalism and Planck-scale / quantum-informational physics. We are scrupulous about status: the *mathematical* statements (e.g. that Top(Ab) is not abelian) are [EST]; the *physical* readings (that this matters for quantum gravity) are [HEU] or [SPEC].

4.1 The C^∞ chart hypothesis at the Planck scale

Remark 4.1 (Operational dissolution of smoothness). Combining the gravitational Schwarzschild radius $r_s \sim GE/c^4$ with the Compton wavelength $\lambda_C \sim \hbar/(Ec)$ of a probe of energy E, one finds that localizing an event to better than the Planck length

$$\ell_P \;=\; \sqrt{\frac{\hbar G}{c^3}} \;\approx\; 1.6 \times 10^{-35}\,\mathrm{m}$$

requires concentrating enough energy in a small enough region to form a black hole, obstructing the measurement. Thus there is no operational procedure that resolves the "C^∞ near-p" structure of (P2)–(P3) below ℓ_P. The smooth chart is an extrapolation, not an observation. [HEU]

The correct methodological response is not to *assume* a replacement (a lattice, a causal set, a spin network) but to ask: *what is the minimal categorical enlargement of "space" that (a) contains all manifolds faithfully, (b) is closed under the operations physics performs (limits, colimits, quotients, function spaces), and (c) supports the homological algebra a quantum theory needs?* Sections 5–6 give the mathematicians' answer to (a)–(c): condensed sets. That this answer is also the right *physical* arena is the program's central [SPEC] hypothesis, developed in Parts VI–VII.

4.2 Manifold locality versus quantum nonlocality

Remark 4.2 (The locality tension). Presupposition (P5) — manifold locality — states that spacelike-separated regions are causally and statistically independent. Quantum mechanics violates the corresponding classical (Bell/CHSH) inequalities: an entangled pair distributed across spacelike separation exhibits correlations

$$|E(a,b) - E(a,b')| + |E(a',b) + E(a',b')| \;\le\; 2 \quad \text{(classical, local)},$$

whereas quantum mechanics attains $2\sqrt{2}$. No signal is transmitted — microcausality (section 2, and Part II) is preserved — yet the *correlational* structure is not that of independent manifold-local subsystems. [HEU]

The manifold encodes "nearness" through open sets and metric distance, a fundamentally *archimedean* and *connected* notion. Entanglement encodes a nearness that is neither: two qubits can be maximally correlated while arbitrarily far apart in g-distance, and the correlation is discrete/algebraic, not metric. The pro-étale site (section 5) is built on *totally disconnected* probe objects, whose "nearness" is combinatorial rather than metric. The [SPEC] suggestion — to be made precise only in Parts III and VI — is that entanglement is more naturally indexed by such totally disconnected, profinite probes than by manifold neighborhoods. We flag this now, but prove nothing about it here.

4.3 The abelian-category obstruction

The third tension is purely mathematical, hence [EST], and it is the historical motivation for condensed mathematics itself. Fields in a quantum theory live in topological vector spaces (of functions, distributions, states); building a quantum theory requires homological algebra (derived functors, resolutions, cohomology) on these objects. But the ambient category is defective:

Proposition 4.3 (Topological abelian groups do not form an abelian category)**.** *Let* $\mathrm{Top}(\mathrm{Ab})$ *denote the category of topological abelian groups with continuous homomorphisms. Then* $\mathrm{Top}(\mathrm{Ab})$ *is additive but* not *abelian.*

Proof. In any abelian category a morphism that is both monic and epic is an isomorphism. Consider the identity map on underlying groups

$$\phi : \mathbb{R}_{\mathrm{disc}} \longrightarrow \mathbb{R},$$

where $\mathbb{R}_{\mathrm{disc}}$ is $\mathbb{R}$ with the discrete topology and $\mathbb{R}$ carries its usual topology. The map ϕ is a continuous homomorphism (every map out of a discrete space is continuous). It is *monic*: it is injective on underlying sets, and precomposition is cancellable. It is *epic*: it is surjective on underlying sets, so postcomposition is cancellable. But ϕ is *not* an isomorphism, because its set-theoretic inverse $\mathbb{R} \to \mathbb{R}_{\mathrm{disc}}$ is not continuous (the usual topology is strictly coarser than the discrete one). Equivalently, in Top(Ab) the map ϕ has zero kernel and zero cokernel, yet is not invertible, so the canonical map $\operatorname{coim} \phi \to \operatorname{im} \phi$ fails to be an isomorphism. Hence Top(Ab) is not abelian. □

Remark 4.4 (Why this matters). Theorem 4.3 is not a curiosity. It means kernels and cokernels of continuous homomorphisms are computed in incompatible ways, so quotients, exact sequences and derived functors misbehave. Any framework that wants to do homological algebra with topological/analytic coefficients — as a quantum field theory or a quantum theory of geometry must — is obstructed at the foundations. The signal achievement of condensed mathematics (theorem 6.7) is that Cond(Ab) *is* a well-behaved abelian category, into which Top(Ab) embeds fully faithfully on compactly generated groups. This is the crisp, [EST] payoff that motivates the entire enterprise, independent of any physical speculation.

5 Profinite Sets, Extremally Disconnected Sets, and the Pro-étale Site of a Point

We now build the machinery. Everything in sections 5 to 7 is [EST], following Scholze's *Lectures on Condensed Mathematics* [1], Bhatt–Scholze [5], and Barwick–Haine [6].

5.1 Profinite sets

Definition 5.1 (Profinite set)**.** A *profinite set* is a topological space satisfying any of the following equivalent conditions:

1. it is compact, Hausdorff and totally disconnected;
2. it is a cofiltered limit $S = \varprojlim_i S_i$ of finite discrete sets S_i in Top;
3. it is homeomorphic to $\operatorname{Spec}(B)$ for a Boolean ring B (Stone duality).

Morphisms are continuous maps. We write ProFin for the resulting category.

Proposition 5.2 (Equivalence of the three descriptions)**.** *The three conditions in theorem 5.1 define the same class of spaces.*

Proof sketch. (2) ⇒ (1): a cofiltered limit of finite discrete (hence compact Hausdorff totally disconnected) spaces is closed in the product $\prod_i S_i$, which is compact Hausdorff by Tychonoff; being closed in a totally disconnected space it is itself compact Hausdorff totally disconnected. (1) ⇒ (2): for a compact Hausdorff totally disconnected S, the clopen partitions of S form a cofiltered system of finite quotients $S \twoheadrightarrow S_i$, and the canonical map

$S \to \varprojlim_i S_i$ is a continuous bijection between compact Hausdorff spaces, hence a homeomorphism. (1) $\Leftrightarrow$ (3) is Stone duality: $B := \{\text{clopen subsets of } S\}$ is a Boolean ring, and $S \cong \operatorname{Spec}(B)$; conversely $\operatorname{Spec}(B)$ of any Boolean ring is compact Hausdorff totally disconnected. $\square$

Example 5.3 (p-adic integers and the Cantor set)**.** $\mathbb{Z}_p = \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$ is profinite: a cofiltered limit of finite discrete rings. The Cantor set $\{0,1\}^{\mathbb{N}} = \varprojlim_n \{0,1\}^n$ is profinite. Both are compact, Hausdorff, totally disconnected — and both are archetypes of "discrete data organized into a limit", exactly the structure a lattice or causal approximation to spacetime wants but which the manifold cannot host natively. The accompanying Haskell code (section 9.3) computes with $\mathbb{Z}/p^n$ truncations of $\mathbb{Z}_p$ as an inverse system.

5.2 Extremally disconnected sets

Definition 5.4 (Extremally disconnected / Stonean)**.** A compact Hausdorff space S is *extremally disconnected* (or *Stonean*) if the closure of every open subset is open. We write ExtDisc $\subseteq$ ProFin for the full subcategory of extremally disconnected profinite sets.

Theorem 5.5 (Gleason: extremally disconnected = projective)**.** *For a profinite (indeed compact Hausdorff) set S, the following are equivalent:*

1. *S is extremally disconnected;*
2. *S is a* projective object *of* CompHaus*: for every surjection $T \twoheadrightarrow T'$ and every map $S \to T'$, there is a lift $S \to T$;*
3. *S is a retract of the Stone–Čech compactification βI of some discrete set I.*

Proof sketch. (1) $\Leftrightarrow$ (2) is Gleason's theorem [7]: extremally disconnected compact Hausdorff spaces are precisely the projectives, because surjections of compact Hausdorff spaces admit continuous sections exactly when the source has the closure-of-open-is-open property. (3) $\Rightarrow$ (1): βI is extremally disconnected (its clopen algebra is the complete Boolean algebra $\mathcal{P}(I)$), and a retract of an extremally disconnected space is extremally disconnected. (1) $\Rightarrow$ (3): by the equivalence (1) $\Leftrightarrow$ (2) just established, S is projective; choosing a set I with a surjection $\beta I \twoheadrightarrow S$ (e.g. $I = S$ as a discrete set, using the counit $\beta(S_{\text{disc}}) \twoheadrightarrow S$), projectivity of S splits it, exhibiting S as a retract of βI. $\square$

Proposition 5.6 (Extremally disconnected sets cover)**.** *Every profinite set S admits a surjection $\widetilde{S} \twoheadrightarrow S$ from an extremally disconnected profinite set $\widetilde{S}$. Consequently* ExtDisc *is a generating class for the pro-étale site.*

Proof. Take $\widetilde{S} = \beta(S_{\text{disc}})$, the Stone–Čech compactification of the underlying set of S made discrete. The identity $S_{\text{disc}} \to S$ extends, by the universal property of β, to a continuous map $\beta(S_{\text{disc}}) \to S$; it is surjective because its image is compact (hence closed) and dense. By theorem 5.5, $\beta(S_{\text{disc}})$ is extremally disconnected. $\square$

5.3 The pro-étale site of a point

Definition 5.7 (Pro-étale site of a point)**.** The *pro-étale site of a point*, denoted $*_{\text{pro-ét}}$, is the category ProFin of profinite sets equipped with the Grothendieck topology whose covering families are the *finite* families $\{S_i \to S\}_{i=1}^n$ that are *jointly surjective*.[1]

This is the site of theorem 2.1 realized on profinite sets. It is the "Spec(k)-specialization" of the general pro-étale topology of Bhatt–Scholze [5]: whereas they refine the étale topology of a scheme to make it locally contractible and to compute ℓ-adic pro-systems honestly, we take the case of a point, where the site is just profinite sets and jointly-surjective covers.

Remark 5.8 (The rhetorical precedent)**.** The pro-étale topology was introduced to *repair specific pathologies* of the naive (étale) topology in algebraic geometry — e.g. to see all lisse ℓ-adic local systems on non-normal schemes without artificial inverse limits. This is precisely the rhetorical shape of the present proposal: a topology finer than the naive one, engineered to repair pathologies. Part I's claim is the exact analogue one scale up: the condensed/pro-étale enlargement repairs pathologies of the naive smooth-manifold structure (theorem 4.3) the way the pro-étale site repairs pathologies of the étale site. That this analogy extends to *physics* is [SPEC]; that it holds in *mathematics* is [EST]. [HEU]

6 Condensed Sets and Condensed Abelian Groups

6.1 Condensed sets

Definition 6.1 (Condensed set)**.** A *condensed set* is a functor $T : \mathrm{ProFin}^{\mathrm{op}} \to \mathrm{Set}$ (a presheaf) satisfying the sheaf conditions for $*_{\text{pro-ét}}$:

1. $T(\varnothing) = *$ (a one-point set);
2. for all profinite S, S', the natural map $T(S \sqcup S') \to T(S) \times T(S')$ is a bijection;
3. for every surjection $S' \twoheadrightarrow S$ of profinite sets,
$$T(S) \longrightarrow T(S') \rightrightarrows T(S' \times_S S')$$
is an equalizer.

Morphisms are natural transformations. The resulting category is Cond(Set).

[1] The restriction to *finite* jointly surjective families is a deliberate definitional choice, not a consequence of quasi-compactness. Arbitrary (infinite) jointly surjective families must *not* be admitted as covers: e.g. the family of all singleton inclusions $\{\,\{x\} \hookrightarrow \mathbb{Z}_p\,\}_{x\in\mathbb{Z}_p}$ is jointly surjective yet has no finite jointly surjective subfamily, and admitting it would refine the topology until every condensed set became discrete, collapsing the theory. A finite jointly surjective family $\{S_i \to S\}_{i=1}^n$ is equivalent, via the single surjection $\coprod_i S_i \twoheadrightarrow S$ (using theorem 6.1(2)), to a covering by one surjection of profinite sets; these are the covers of $*_{\text{pro-ét}}$. That the resulting topos agrees with the point-specialization of the general Bhatt–Scholze pro-étale topology [5], and may be presented equivalently on any of ProFin, CompHaus, or ExtDisc, is a theorem of Scholze [1] (theorem 6.2), depending only on the generated Grothendieck topology (section 2).

There is a genuine set-theoretic subtlety: ProFin is not small, so one fixes an uncountable strong limit cardinal κ, works with κ-condensed sets (sheaves on κ-small profinite sets), and takes the union over all κ. We suppress κ throughout; it does not affect any statement below. The pyknotic variant of Barwick–Haine [6] makes the opposite set-theoretic choice (fix a universe, sheaves on *all* compact Hausdorff spaces) and is essentially equivalent; we note the distinction but do not depend on it.

Theorem 6.2 (Presentation by extremally disconnected sets)**.** *Restriction along* $\mathrm{ExtDisc} \hookrightarrow \mathrm{ProFin}$ *induces an equivalence between* $\mathrm{Cond}(\mathrm{Set})$ *and the category of functors* $\mathrm{ExtDisc}^{\mathrm{op}} \to \mathrm{Set}$ *that send finite disjoint unions to finite products. Equivalently, a condensed set is determined by its values on extremally disconnected profinite sets, where the only sheaf condition remaining is preservation of finite products.*

Proof sketch. By theorem 5.6 every profinite set is covered by an extremally disconnected one, so ExtDisc generates the topology and $\mathrm{Sh}(*_{\text{pro-ét}}) \simeq \mathrm{Sh}(\mathrm{ExtDisc})$ for the induced topology (sheaf theory depends only on the generated topology, section 2). On ExtDisc the covering surjections $S' \twoheadrightarrow S$ *split* — because the target S, being extremally disconnected, is projective (theorem 5.5) — so the equalizer condition (3) of theorem 6.1 is automatically satisfied once (1)–(2), i.e. finite-product preservation, hold. Hence a condensed set is exactly a finite-product-preserving presheaf on ExtDisc. □

Theorem 6.2 is the computational heart of the theory: it reduces the sheaf condition to a finite-product condition on projective probes. Our Haskell probe model (section 9.3) exploits exactly this: a condensed set is modeled by its "probe function" on finite approximations to extremally disconnected sets.

6.2 The comparison functor from topological spaces

Definition 6.3 (The functor $\underline{(-)} : \mathrm{Top} \to \mathrm{Cond}(\mathrm{Set})$)**.** For a topological space X, define the condensed set $\underline{X}$ by

$$\underline{X}(S) \;=\; \mathrm{Cont}(S, X) \;=\; \{\text{continuous maps } S \to X\}, \qquad S \in \mathrm{ProFin}.$$

Functoriality in S is by precomposition; functoriality in X is by postcomposition.

Theorem 6.4 (Scholze, Prop. 1.7: the comparison functor)**.** *The functor* $\underline{(-)} : \mathrm{Top} \to \mathrm{Cond}(\mathrm{Set})$ *of theorem 6.3 is well-defined (each* $\underline{X}$ *is a condensed set) and* faithful*. Its restriction to compactly generated weak Hausdorff spaces is* fully faithful*. It admits a left adjoint* $T \mapsto T(*)_{\mathrm{top}}$ *sending a condensed set* T *to the set* $T(*)$ *equipped with the quotient topology from* $\coprod_{S \to T} S \twoheadrightarrow T(*)$*.*

Proof sketch. That $\underline{X}$ is a condensed set: $\mathrm{Cont}(-, X)$ sends $\varnothing \mapsto *$ and disjoint unions to products, and for a surjection $S' \twoheadrightarrow S$ of profinite sets a continuous map $S \to X$ is the same as a continuous map $S' \to X$ equalizing the two projections $S' \times_S S' \rightrightarrows S'$, because $S' \twoheadrightarrow S$ is a quotient map of compact Hausdorff spaces; this is the equalizer condition. Faithfulness: a continuous $f : X \to Y$ is determined by its action on points, and points are captured by $S = *$, so $\underline{f} = \underline{g}$ forces $f = g$. Full faithfulness on compactly generated weak Hausdorff spaces: a natural transformation $\underline{X} \to \underline{Y}$ is in particular a map on $S = *$ (a function on points) that is compatible with all continuous probes $S \to X$; for compactly generated X the topology is determined by such probes (indeed by maps from compact Hausdorff, equivalently profinite, sets), so the underlying function is continuous and the transformation is $\underline{f}$ for a unique continuous f. The left adjoint is the evident "underlying topological space" construction; adjunction is a direct check on universal properties. Full details are in [1, Lecture 1]. □

Corollary 6.5 (Underlying topological spaces of manifolds embed fully faithfully)**.** *Let* $U : \mathrm{Mfd} \to \mathrm{CGWH}$ *be the underlying-topological-space functor. Then:*

1. *The composite* $\mathrm{CGWH} \xrightarrow{\underline{(-)}} \mathrm{Cond(Set)}$ *is fully faithful; hence the underlying topological space of every smooth manifold* M *is recovered, together with all its topological structure, from the condensed set* $\underline{M}$*, and* all *continuous maps between (the underlying spaces of) manifolds correspond bijectively to condensed maps* $\underline{M} \to \underline{N}$*.*

2. *The composite* $\mathrm{Mfd} \xrightarrow{U} \mathrm{CGWH} \xrightarrow{\underline{(-)}} \mathrm{Cond(Set)}$ *is* faithful but not full*: condensed maps* $\underline{M} \to \underline{N}$ *recover the continuous maps* $M \to N$*, which strictly contain the smooth maps whenever* $\dim M \geq 1$ *and* $\dim N \geq 1$*. Thus the smooth structure is extra data that* $\underline{(-)}$ *does not encode.*

Proof. A smooth manifold is second countable, Hausdorff and locally Euclidean, hence compactly generated and weak Hausdorff, so U is well-defined; part (1) is then the fully-faithful clause of theorem 6.4 applied to underlying spaces. For part (2), theorem 6.4 gives $\mathrm{Hom}_{\mathrm{Cond(Set)}}(\underline{M}, \underline{N}) = \mathrm{Cont}(M, N)$, the set of *all* continuous maps; but $\mathrm{Hom}_{\mathrm{Mfd}}(M, N) = C^\infty(M, N) \subsetneq \mathrm{Cont}(M, N)$ whenever $\dim M \geq 1$ and $\dim N \geq 1$ (there exist continuous non-smooth maps, e.g. $t \mapsto |t|$, $\mathbb{R} \to \mathbb{R}$). (In the edge case $\dim M = 0$, i.e. M discrete, every continuous map is smooth and the two Hom-sets coincide.) Hence the functor is not full. It is faithful because a smooth (indeed continuous) map is determined by its values on points, captured by $S = *$. □

Theorem 6.5 is the precise, honest version of "manifolds pass to condensed spaces." The manifold's *topology* is not discarded: its underlying space sits inside Cond(Set) as a particularly tame object, fully faithfully. The *smooth* structure, by contrast, is extra data that the topological comparison functor does not encode — recovering geometry (a smooth or metric structure) from condensed/algebraic data is a separate problem, deferred to Part V. Either way the ambient condensed category has strictly more room (theorem 8.1).

6.3 Condensed abelian groups and the abelian-category repair

Definition 6.6 (Condensed abelian group)**.** A *condensed abelian group* is an abelian-group object in Cond(Set); equivalently, a sheaf of abelian groups on $*_{\text{pro-ét}}$. The category is Cond(Ab). More generally, for a ring R, condensed R-modules are sheaves of R-modules on $*_{\text{pro-ét}}$.

Theorem 6.7 (Clausen–Scholze: Cond(Ab) is abelian)**.** *The category* Cond(Ab) *is an abelian category. It is complete and cocomplete, has enough projectives (the free condensed abelian groups* $\mathbb{Z}[S]$ *on extremally disconnected* S*) and enough injectives, and the forgetful functor* $\mathrm{Top(Ab)} \to \mathrm{Cond(Ab)}$, $A \mapsto \underline{A}$, *is fully faithful on compactly generated topological abelian groups and exact where defined. In particular, kernels and cokernels of maps of condensed abelian groups are computed* objectwise *and are again condensed.*

Proof sketch. $\mathrm{Cond(Ab)} = \mathrm{Sh}(*_{\text{pro-ét}}, \mathrm{Ab})$ is a category of abelian-group-valued sheaves on a site, and any such category is abelian, complete and cocomplete, with objectwise kernels and (sheafified) cokernels; enough injectives is Grothendieck's theorem for sheaf categories. By theorem 6.2 a condensed abelian group is a finite-product-preserving functor on ExtDisc, on which covers split, so cokernels are already objectwise (no sheafification needed) — this is exactly the mechanism that repairs theorem 4.3. Free condensed abelian groups $\mathbb{Z}[S]$ on extremally disconnected S are projective and generate, giving enough projectives. Full faithfulness on compactly generated groups follows from theorem 6.4 applied to abelian-group objects. See [1, Lectures 1–2] and [2]. □

Corollary 6.8 (Repair of the pathology of theorem 4.3)**.** *The map* $\phi : \mathbb{R}_{\mathrm{disc}} \to \mathbb{R}$ *of theorem 4.3, viewed via* $\underline{(-)}$ *as a map* $\underline{\mathbb{R}_{\mathrm{disc}}} \to \underline{\mathbb{R}}$ *in* Cond(Ab)*, has a nonzero cokernel* $Q := \mathrm{coker}(\phi)$ *that is a genuine condensed abelian group. Thus the failure of* ϕ *to be an isomorphism is recorded honestly by a nontrivial object* Q*, and* $0 \to \underline{\mathbb{R}_{\mathrm{disc}}} \to \underline{\mathbb{R}} \to Q \to 0$ *is a short exact sequence in the abelian category* Cond(Ab)*.*

Proof. In Cond(Ab) the cokernel of ϕ exists (theorem 6.7) and is nonzero precisely because ϕ is not an isomorphism of condensed groups: the two topologies on $\mathbb{R}$ give genuinely different condensed objects (their probe-values $\mathrm{Cont}(S, \mathbb{R}_{\mathrm{disc}})$ and $\mathrm{Cont}(S, \mathbb{R})$ differ already for S a convergent sequence, since a convergent sequence has few maps to a discrete space but many to $\mathbb{R}$). Exactness of the displayed sequence is then the definition of Q as cokernel together with injectivity of ϕ (its kernel is 0 because it is objectwise injective). □

Theorem 6.8 is the whole point in miniature: the very map that *broke* Top(Ab) (theorem 4.3) becomes an unremarkable short exact sequence in Cond(Ab). This is established mathematics ([EST]), and it is the model for what the program hopes condensed mathematics will do for the pathologies of manifold-based physics.

Remark 6.9 (Categorical caveats, stated honestly)*.* Two caveats belong here, both [EST]. First, Cond(Set) is *not* a Grothendieck topos: it is a locally small, locally cartesian closed *infinitary pretopos* lacking a small generator and a subobject classifier. It is Cond(Ab), not Cond(Set), that is the well-behaved *abelian* category. Second, the pyknotic sets of Barwick–Haine [6] form a coherent topos while condensed sets form only a pretopos — a real, if largely foundational, distinction driven purely by the treatment of set-theoretic size. Neither caveat affects the physical program, but honesty (and section 1.2) requires naming them.

7 Sheaves, Descent, and Space from Local Data

Having built the site and its sheaf categories, we make explicit the mechanism by which "space" is assembled from local data — the mechanism Part II will specialize to observables.

Definition 7.1 (Condensed anima (informal))**.** Replacing Set by the ∞-category of anima (homotopy types / ∞-groupoids) in theorem 6.1 yields *condensed anima*: sheaves of spaces on $*_{\text{pro-ét}}$. These form the natural ∞-categorical home for a "condensed homotopy type," and are the ambient objects Parts III–VI will use. We use only the 1-categorical Cond(Set) and Cond(Ab) in Part I and mention condensed anima only for forward reference. [EST]

Proposition 7.2 (Descent for condensed sets)**.** *Let $\{S_i \to S\}_{i=1}^n$ be a jointly surjective finite family of profinite sets and T a condensed set. Then the value $T(S)$ is computed as the equalizer*

$$T(S) \;=\; \mathrm{eq}\left(\prod_i T(S_i) \rightrightarrows \prod_{i,j} T(S_i \times_S S_j)\right).$$

That is, a global element over S is exactly a compatible family of local elements over the S_i.

Proof. This is the sheaf condition (theorems 2.2 and 6.1) for the covering family $\{S_i \to S\}$, using that a finite jointly surjective family is a cover in $*_{\text{pro-ét}}$ and that finite covers reduce (by theorem 6.1(2)) to the single surjection $\coprod_i S_i \twoheadrightarrow S$. $\square$

Example 7.3 (Gluing a condensed set from a two-element cover)**.** Let $S = \{0, 1, 2\}$ (finite discrete, hence profinite) and cover it by $S_1 = \{0, 1\}$ and $S_2 = \{1, 2\}$ via the inclusions. Then $S_1 \times_S S_2 = \{1\}$. For a condensed set T, theorem 7.2 says

$$T(\{0, 1, 2\}) = \{(a, b) \in T(S_1) \times T(S_2) : a|_{\{1\}} = b|_{\{1\}}\}.$$

A global section is a pair of local sections agreeing on the overlap $\{1\}$. The accompanying Haskell code (section 9.3) implements exactly this pullback/equalizer as a runnable "sheaf-gluing" demonstration, verifying associativity of gluing on a three-chart cover.

Remark 7.4 (Space as descent data — the physical reading). Theorems 7.2 and 7.3 make precise the slogan of theorem 2.3: in the condensed world, "space" is not a bag of pre-existing points but a rule assigning compatible data to probes and gluing them. This is the structural reason the condensed picture is attractive for a background-independent physics: it never posits (P1) a completed point-set in advance; points are recovered, when they exist, as $T(*)$, but the primitive datum is the probe-indexed descent structure. That this reading is the *correct physical ontology* is [SPEC] (Parts VI–VII); that descent is the mechanism assembling condensed spaces is [EST].

8 The Comparison: Manifold $\longrightarrow$ Condensed Space

This section is the paper's central deliverable: an explicit, side-by-side comparison of the manifold and condensed notions of space, and a commutative diagram locating manifolds inside the condensed world.

8.1 The comparison, feature by feature

Table 1 tabulates the two notions against the presuppositions (P1)–(P5) of theorem 3.3 and against the homological criterion of section 4.3. The manifold column is [EST] classical geometry; the "what condensed buys" column mixes [EST] mathematics with clearly-labelled [HEU]/[SPEC] physical readings.

$$
\begin{tikzcd}
\mathrm{Fin} \arrow[r, hook] \arrow[d, hook] & \mathrm{ExtDisc} \arrow[r, hook] \arrow[d, hook] & \mathrm{ProFin} \arrow[d, hook] \arrow[dr, hook, "\text{Yoneda}"] & \\
\mathrm{Mfd} \arrow[r, "U"'] & \mathrm{CGWH} \arrow[r, hook] & \mathrm{Top} \arrow[r, "\underline{(-)}"'] & \mathrm{Cond(Set)} \\
& & & \mathrm{Cond(Ab)} \arrow[u, "\text{forget}"']
\end{tikzcd}
$$

Figure 1: The condensed enlargement. *Hooked* arrows are fully faithful (full-subcategory inclusions, and the Yoneda embedding ProFin $\hookrightarrow$ Cond(Set)). The three *plain* arrows are only faithful: the underlying-space functor $U : \mathrm{Mfd} \to \mathrm{CGWH}$ forgets smoothness (smooth maps $\subsetneq$ continuous maps, theorem 6.5); $\underline{(-)} : \mathrm{Top} \to \mathrm{Cond(Set)}$ is fully faithful only on the CGWH subcategory (theorem 6.4); and the forgetful functor Cond(Ab) $\to$ Cond(Set) is faithful but not full. Finite sets sit inside extremally disconnected profinite sets, which generate the site (theorem 6.2); the middle vertical arrow uses that every extremally disconnected profinite set is compact Hausdorff, hence CGWH. Manifolds — compactly generated and weak Hausdorff, though *not* in general compact — thus have their underlying *topological* spaces fully faithfully embedded via CGWH $\hookrightarrow$ Cond(Set), while their smooth structure is extra data. Condensed abelian groups supply (theorem 6.7) the abelian category Top(Ab) failed to be.

8.2 The commutative diagram

Figure 1 is the geometric content of table 1: it exhibits Cond(Set) as a common home for the discrete/profinite world (top row) and the topological/manifold world (bottom row), with the underlying topological spaces of manifolds fully faithfully embedded (and Mfd itself faithfully) and condensed abelian groups supplying the missing abelian structure. The diagram commutes: the two ways of sending a finite set into Cond(Set) (through ProFin and through Top) agree, because a finite set's representable sheaf equals the condensed set of the finite discrete space.

8.3 Why condensed is more flexible: the precise claim

Proposition 8.1 (Strict enlargement)**.** *The essential image of* $\underline{(-)} : \mathrm{Top} \to \mathrm{Cond(Set)}$ *is a* proper *subcategory: there exist condensed sets not isomorphic to* $\underline{X}$ *for any topological space* X*. Moreover* Cond(Set) *admits all small limits and colimits and is cartesian closed, whereas* Top *and* Mfd *do not have well-behaved quotients or internal homs in general.*

Proof sketch. For strictness, the free condensed abelian group $\mathbb{Z}[S]$ on an infinite profinite set S (or, at the level of sets, the quotient $\underline{\mathbb{R}}/\underline{\mathbb{R}_{\mathrm{disc}}}$ of theorem 6.8) is a condensed object whose value on probes does not arise as $\mathrm{Cont}(-, X)$ for any single topological space X; concretely, $\mathbb{Z}[S](*)$ is the free abelian group on the points of S while its higher probe values encode the profinite topology, a combination no representable $\underline{X}$ realizes. Completeness, cocompleteness and cartesian closedness of Cond(Set) hold because it is a category of sheaves (a pretopos) on a site; that Top lacks these is classical (quotients of topological spaces need not be well-behaved; Top is not cartesian closed without restricting to compactly generated spaces). □

Remark 8.2 (The honest summary). Theorem 8.1 together with theorems 6.5 and 6.7 is the complete, honest statement of Part I's thesis at the level of established mathematics:

> *The underlying topological spaces of manifolds embed fully faithfully into the strictly larger category of condensed sets (and* Mfd *itself embeds faithfully, its smooth structure being extra data); that category is complete, cocomplete and cartesian closed, and its abelian-group objects form a genuine abelian category repairing the defect of topological abelian groups.*

This is entirely [EST]. The *physical* thesis — that the extra room is where Planck-scale discreteness and quantum nonlocality live, and that spacetime is fundamentally condensed — is [SPEC], and is the subject of Parts VI–VII. We do not conflate the two.

9 Results

We collect and slightly extend the main structural results, and record the Haskell verification.

9.1 Main structural theorem

Theorem 9.1 (Structure of the condensed enlargement of space)**.** *Let* $\underline{(-)} : \mathrm{Top} \to \mathrm{Cond(Set)}$ *be the comparison functor of theorem 6.3. Then:*

1. (Faithful embedding) $\underline{(-)}$ *is faithful, and fully faithful on compactly generated weak Hausdorff spaces; hence the underlying-space functor* $\mathrm{CGWH} \hookrightarrow \mathrm{Cond(Set)}$ *is fully faithful, while* $\mathrm{Mfd} \to \mathrm{Cond(Set)}$ *is faithful but not full* (theorem 6.5)*.*

2. (Presentation) Cond(Set) *is equivalent to finite-product-preserving presheaves on* ExtDisc (theorem 6.2)*; the sheaf condition reduces to a product condition on projective probes.*

3. (Abelian repair) Cond(Ab) *is an abelian category with enough projectives and injectives, into which compactly generated topological abelian groups embed fully faithfully; the non-isomorphism* $\mathbb{R}_{\mathrm{disc}} \to \mathbb{R}$ *that obstructs abelianness of* Top(Ab) *becomes a short exact sequence* (theorems 4.3, 6.7 and 6.8)*.*

4. (Strictness and closure) *the embedding is proper, and* Cond(Set) *is complete, cocomplete and cartesian closed* (theorem 8.1)*.*

Consequently the condensed world is a common refinement of the profinite/discrete and the topological/manifold worlds, strictly larger than either, on which the homological algebra required by a quantum theory of geometry is available.

Proof. Each clause is proved in the section indicated: (1) is theorems 6.4 and 6.5; (2) is theorem 6.2; (3) is theorems 4.3, 6.7 and 6.8; (4) is theorem 8.1. The final sentence assembles them: (1) gives faithfulness of the manifold picture, (4) gives strict enlargement and categorical closure, (3) gives the homological algebra, and (2) makes all of it computable on projective probes. □

Theorem 9.1 is the formal statement of the arrow Manifold $\to$ Condensed Space promised in the abstract. It is [EST] in its entirety.

9.2 A reconstruction lemma: points from probes

Lemma 9.2 (Points are recovered as $*$-sections)**.** *For any topological space X, the underlying point-set of X is $\underline{X}(*) = \mathrm{Cont}(*, X)$, naturally in X. For a general condensed set T, the set $T(*)$ is the "set of points" of T, and the left adjoint of theorem 6.4 endows it with the finest topology making all probes $S \to T$ continuous.*

Proof. $\mathrm{Cont}(*, X) = X$ as sets, functorially. For general T, $T(*)$ is the value on the terminal profinite set; the left adjoint's construction (theorem 6.4) is exactly the quotient topology from $\coprod_{S \to T} S \twoheadrightarrow T(*)$. □

Theorem 9.2 is the precise sense in which the condensed picture *derives* points rather than presupposing them (P1): a condensed set can have any set of points, including none in exotic cases, while carrying rich probe structure. This is the structural room referenced in table 1 and theorem 7.4.

9.3 Haskell formalization

To make the abstractions concrete and machine-checkable, the paper is accompanied by a small Haskell package (module `Main` plus supporting modules `Profinite`, `Condensed`, `Sheaf`). It realizes three of the paper's constructions:

1. **Profinite sets as inverse limits** (theorems 5.1 and 5.3): the inverse system $\mathbb{Z}/p^n\mathbb{Z}$ with the transition maps $\mathbb{Z}/p^{n+1} \to \mathbb{Z}/p^n$, together with finite truncations of coherent sequences, modeling $\mathbb{Z}_p$ as $\varprojlim_n \mathbb{Z}/p^n$.

2. **Condensed-set probes** (theorems 6.1 and 6.2): a condensed set represented by its probe function on finite approximants to extremally disconnected sets, with the finite-product (disjoint-union $\mapsto$ product) law of theorem 6.2 checked on examples.

3. **Sheaf gluing** (theorems 7.2 and 7.3): the equalizer/pullback that glues local sections agreeing on overlaps, verified on the three-chart cover of theorem 7.3 together with an associativity check on a four-chart refinement.

The `main` function runs all three demonstrations and prints the verified equalities. The code compiles under GHC with no external dependencies. Its role is illustrative and verificational, not foundational: it exhibits the finite, computable shadow of the (generally infinite) condensed constructions.

10 Discussion

10.1 Explicit non-identifications

Intellectual honesty (and the program's discipline, section 1.2) requires distinguishing this proposal from neighboring sheaf-/topos-theoretic approaches to physics with which it must *not* be silently merged.

- **Cohesive homotopy type theory** (Schreiber [16]) is a *synthetic*, internal-language route to smooth ∞-groupoids via shape/flat/sharp modalities. Condensed mathematics is an *analytic*, external route via sheaves on extremally disconnected sets. Both aim near the target of Part I/IV, but they are technically different and should be compared, not conflated. [EST] (as a statement about the two frameworks).
- **Topos quantum theory** (Isham–Döring [17]) uses presheaves on a context category of commutative subalgebras to reformulate quantum *logic*. This program uses the pro-étale site to reformulate *spacetime and fields themselves*, not primarily quantum logic. Distinct aims, distinct sites. [EST].
- **Causal sets** (Bombelli–Lee–Meyer–Sorkin [15]) discretize spacetime as a locally finite partial order ("order + number = geometry"). This is a *related but technically distinct* discretization precedent; we cite it as structural analogy, not equivalence. Claiming a condensed set *is* a causal set, or vice versa, would require a new argument we do not make. [HEU].
- **Noncommutative geometry** (Connes [13]) reconstructs a Riemannian spin manifold from a commutative spectral triple. This is the nearest *algebraization* precedent (and the central anchor of Part V), but it is Riemannian/static and reconstructs a manifold, whereas the condensed proposal enlarges the category of spaces themselves. Related, not identical. [HEU]/[EST].

10.2 Limitations and status composition

We restate, per section 1.2, the epistemic bookkeeping for Part I in the form of the program's summary row (cf. table 2).

The limitations are explicit:

1. *Nothing here shows spacetime is condensed.* We have shown that the category of spaces *can* be enlarged, faithfully and profitably, and that the enlargement repairs a genuine homological defect. Whether physics *uses* this room is [SPEC].
2. *No dynamics.* Part I is kinematic. There is no Einstein equation, no action, no evolution. Dynamics enters (speculatively) only in Parts IV–VII, and any claim that gravity or field equations *emerge* from condensed gluing is capped at [SPEC] by worst-component composition.
3. *The metric and smooth structure are not recovered by* $\underline{(-)}$. The comparison functor preserves the underlying topology, not the Lorentzian metric or the C^∞ atlas. Recovering geometry (a metric, a causal structure) from condensed/algebraic data is the open problem of Part V, and in the Lorentzian case is open even in the noncommutative-geometry precedent [14].

4. *No existing literature* combines condensed mathematics with quantum gravity; a targeted 2026 search confirms this. This is stated as a feature (unoccupied territory) but forces every physical claim downstream to be presented as a labelled research proposal, not a result.

10.3 What Part I hands to Parts II–VII

Part I fixes the ambient category. Concretely it hands downstream:

- to **Part II** (*Observables as Condensed Sheaves*): the site $*_{\text{pro-ét}}$ and the descent mechanism (theorem 7.2) on which an assignment $U \mapsto \mathrm{Obs}(U)$ of observables becomes a condensed sheaf (or cosheaf/factorization algebra);
- to **Part III** (*Condensed Cohomology and Quantum Information*): the abelian category Cond(Ab) (theorem 6.7) in which condensed cohomology, and the proposed "entanglement as sheaf compatibility," will be defined;
- to **Part IV** (*Fields Without Background Manifolds*): the condensed base C on which fields $\varphi : C \to \mathrm{CondVect}$ will replace $\varphi : M \to \mathbb{R}$, using solid/liquid refinements of Cond(Ab);
- to **Part V** (*Geometry as a Realization Functor*): the target category enlargement for a conjectural condensed-Tannakian reconstruction of geometry;
- to **Parts VI–VII**: the foundational claim, and the discipline that its physical payload is [SPEC].

11 Conclusion

We have argued, at the level of established mathematics, that the smooth manifold is not the most flexible available notion of space, and that the condensed sets of Clausen and Scholze provide a strictly larger, better-behaved category into which the underlying topological spaces of manifolds embed fully faithfully — and Mfd itself faithfully, its smooth structure being extra data (theorems 6.5 and 9.1). We recalled precisely what the manifold formalism presupposes (theorem 3.3), isolated three structural tensions — the operational dissolution of C^∞ charts at the Planck scale, manifold locality versus quantum nonlocality, and the failure of Top(Ab) to be abelian (theorem 4.3) — and showed that the third is *repaired* outright by Cond(Ab) (theorems 6.7 and 6.8). We developed the requisite machinery from scratch: profinite and extremally disconnected sets (theorems 5.1 and 5.5), the pro-étale site of a point (theorem 5.7), condensed sets and their presentation on projective probes (theorems 6.1 and 6.2), and descent as "space from local data" (theorem 7.2). The central deliverable — the comparison Manifold → Condensed Space — is recorded in table 1 and fig. 1, and formalized in theorem 9.1, with an accompanying Haskell realization (section 9.3).

We have been scrupulous about status. The mathematics is [EST]; the proposal that Planck-scale spacetime is fundamentally condensed rather than smooth is [SPEC], clearly bounded and forward-referenced to Parts VI–VII. This is the first, most conservative paper of the series — it commits only to established category theory and condensed mathematics, and to the honest observation that they offer more room than a manifold. The genuinely speculative physics is deferred, by design, to later Parts, where it will be presented as labelled research proposals and where the worst-component composition discipline (section 1.2) will cap every cross-Part claim at the status of its weakest link.

Part I is the foundation. On it, Part II builds observables, Part III information, Part IV fields, Part V geometry, Part VI a synthesis, and Part VII a capstone research program — each composing modularly on the last, and each inheriting from here the ambient category of condensed spaces and the discipline that keeps its mathematics and its speculation apart.

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Feature	**Smooth manifold** (M, g)	**Condensed space** $T \in \mathrm{Cond}(\mathrm{Set})$
Primitive datum	completed point-set + atlas (P1)	probe-indexed descent data (theorem 7.2)
Local model	open $U \subseteq \mathbb{R}^n$ (P2), archimedean	profinite/extremally disconnected sets, totally disconnected
Smoothness	C^∞ transition maps (P3)	none required; smoothness is optional extra data
Separation	Hausdorff, second countable (P4)	no separation axiom imposed; non-separated objects allowed
Points	primitive; continuum many	derived: $T(*)$, may be empty or exotic
Locality	metric neighborhoods, light cones (P5)	combinatorial/algebraic; totally disconnected probes
Category	Mfd: lacks general colimits (not cocomplete), quotients pathological	Cond(Set): complete, cocomplete, cartesian closed
Homological algebra	Top(Ab) *not* abelian (theorem 4.3)	Cond(Ab) *abelian* (theorem 6.7)
Embedding	—	underlying space fully faithful; Mfd faithful, not full (theorem 6.5)

Table 1: Manifold versus condensed space. Left column: [EST] classical differential geometry. Right column: [EST] condensed mathematics. The claim that the right column's extra room is where Planck-scale/nonlocal physics lives is [SPEC] (section 4, Parts VI–VII).

Part	**Established mathematical core**	**Speculative physical claim**	**Status**
I	Condensed sets, pro-étale site, extremally disconnected sets, Top $\to$ Cond faithful (fully faithful on CGWH), Cond(Ab) abelian (theorem 9.1)	Planck-scale spacetime is condensed-set-like; nonlocality accommodated by profinite indexing	[HEU] / [SPEC]

Table 2: Part I epistemic census. The mathematical core is [EST]; the physical extrapolation is [HEU]/[SPEC].