Observables as Condensed Sheaves

Part II of *Toward a Condensed Representation Theory of Physics*

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Abstract

This is the second paper of a modular, seven-part program built on the governing perspective that *physics is the study of realizations of condensed mathematical structures*. Part I embedded smooth spacetime faithfully into the strictly larger, better-behaved category of condensed sets and argued that Planck-scale and quantum-informational phenomena may live in the extra room. Here we take the decisive methodological step of the program: we develop *observables before geometry*. Rather than assuming a background manifold and assigning algebras to its regions, we treat a measurement as a functor, a family of compatible measurements as a sheaf, and geometry as something to be *reconstructed* from the descent data of that sheaf — inverting the usual order *geometry* $\to$ *observables* into *measurement* $\to$ *relations* $\to$ *geometry*. We make this precise. We define a small site of measurement contexts, a condensed sheaf of observables on it, and its dual precosheaf (net) of operator algebras; we prove the descent (gluing) theorem that compatible local observations glue uniquely, and the automorphism-retention theorem that gluing on the associated prestack preserves gauge/automorphism data rather than collapsing to a coarse quotient ($\underline{\mathrm{Aut}}_{[X/G]}(x) \cong \mathrm{Stab}_G(x)$). We show why the naive presheaf of observable values must be sheafified and, more, *derived*: local observations that are pairwise compatible can still fail to globalize, the obstruction being a class in $\check{H}^1$, so the correct observable attached to a context is the derived complex $R\Gamma(U, \mathcal{O})$, not merely its degree-zero part. We position the Haag–Kastler net, the Brunetti–Fredenhagen–Verch locally covariant functor, and the Costello–Gwilliam factorization algebra as the established precedents for this move, address the sheaf-versus-cosheaf fork between them explicitly, and formulate — as this Part's own open, theorem-shaped research question — the specialization by which a condensed sheaf of observables recovers an ordinary net. Throughout we hold to a strict epistemic discipline: sheaf theory, condensed mathematics, algebraic quantum field theory, and Gelfand duality are *established* mathematics ([EST]); the claim that fundamental observables are condensed-sheaf-valued *prior* to geometry, and that geometry is their reconstruction, is a labelled hypothesis ([HEU]/[SPEC]) whose

discharge we forward to Parts V–VII. Runnable Haskell accompanies the paper: a sheaf of observables with a descent-equalizer checker, a Čech obstruction to gluing, and a toy net with verified isotony and locality.

Contents

1 Introduction

1.1 The modular program and its governing perspective

This is Part II of a modular, hierarchical seven-part program organized around a single sentence:

Physics is the study of realizations of condensed mathematical structures.

The program is *modular* in a precise sense: each Part isolates one structural layer, each Part builds only on the layers below it, and the composite claims of the program are never asserted at a warrant stronger than the weakest layer they depend on. Part I [1] supplied the bottom layer. It recalled exactly what the smooth-manifold model of spacetime presupposes, isolated three structural tensions with Planck-scale and quantum-informational physics, and then embedded topological spaces faithfully into the category Cond(Set) of *condensed sets* of Clausen and Scholze [2, 3], via the fully faithful functor $\mathrm{Top}_{\mathrm{cg}} \hookrightarrow \mathrm{Cond(Set)}$ on compactly generated spaces. The upshot of Part I is a working slogan we now take as given: *a "space" is a sheaf on the pro-étale site of a point*, and manifolds are a small, well-behaved corner of a much larger category.

Part II supplies the next layer up: the *observable* layer. Where Part I asked "what is the arena?", Part II asks "what is a measurement, and in what order do measurement and geometry come?" The modular arc is explicit. Part I gives condensed *spaces*; Part II puts a condensed sheaf of *observables* on top of them, but — and this is the point of the Part — in a way that does *not* require the space to be fixed first. Part III [34] will read the resulting gluing/compatibility data informationally; Part V will ask whether geometry can be *reconstructed* as a realization functor applied to exactly the observable data assembled here. Part II is thus the hinge: it is where the program commits to developing observables *before* geometry.

1.2 The methodological inversion: observables before geometry

The default grammar of mathematical physics is *geometry first*. One fixes a manifold M, or a globally hyperbolic spacetime, and *then* assigns to each region $R \subseteq M$ an algebra of observables $\mathfrak{A}(R)$. Locality, causality, and covariance are stated as constraints *on that assignment*, relative to the pre-existing causal and differentiable structure of M. This is the grammar of algebraic quantum field theory (AQFT) in its original Haag–Kastler form [4], and it is enormously successful.

We propose to read the same mathematics in the opposite order. Operationally, one never has access to a manifold. One has access to *measurements*: outcomes, their statistics, and the relations of compatibility, refinement, and joint performability among them. A "region of spacetime" is, from the operational point of view, a *derived* notion — a label for a class of measurements that hang together in a particular way. The methodological inversion of this Part is to take that seriously and build the formalism in the operational order:

$$\boxed{\text{measurement} \;\longrightarrow\; \text{relations} \;\longrightarrow\; \text{geometry}} \tag{1}$$

Measurements come first; the *relations* among them (compatibility, refinement, descent) come second and are what a sheaf records; *geometry* comes third, as something reconstructed from those relations rather than presupposed. Section 8 makes (1) a precise diagram and assigns each arrow an epistemic status.

Two remarks fix the intended scope immediately. First, this inversion is not a rejection of the geometry-first formalism; it is a *re-foundation* of the same equations on operational primitives, in the hope that the extra room in the condensed world (Part I) is exactly where the reconstruction step can breathe. Second, the inversion has genuine, rigorous precedent: Gelfand duality already reconstructs a compact Hausdorff space from its commutative C^*-algebra of observables (§7); Tomita–Takesaki modular theory and the Bisognano–Wichmann theorem already recover geometric data (boosts, causal complements) from an algebra and a state [9, 10, 11]; and the Costello–Gwilliam factorization algebra already assembles global observables from local ones by a colimit [6]. What is *new* here is only the proposal to run this reconstruction in the condensed setting and to treat it as foundational rather than incidental — and that novelty is labelled speculative wherever it does physical work beyond what the precedents prove.

1.3 The epistemic discipline: EST, HEU, SPEC

We inherit, verbatim in spirit, the status calculus of the seed projects and of Part I. Every substantive assertion is tagged:

- [EST] — *established*: rigorous, citable mathematics, or a rigorously proved physics-adjacent theorem. Sheaf theory and descent [16, 17, 19]; condensed mathematics [2]; the Haag–Kastler and Brunetti–Fredenhagen–Verch axioms and their consequences [4, 5]; Costello–Gwilliam factorization algebras and the Gwilliam–Rejzner comparison [6, 7]; Gelfand–Naimark duality [8]; Tomita–Takesaki/Bisognano–Wichmann [9, 10] are all [EST].

- [HEU] — *heuristic*: a physically motivated dictionary entry or translation that is not itself a theorem. The arrow "relations $\to$ geometry" as a *general* mechanism is [HEU].
- [SPEC] — *speculative*: an ontological hypothesis of this program. "Fundamental observables are condensed-sheaf-valued objects *prior* to any geometric structure" is [SPEC].

The calculus is monotone and worst-component-wins: no chain of reasoning is stronger than its weakest link [34]. In particular, however rigorous the sheaf-theoretic and condensed substrate of this Part is, the moment we assert that geometry *emerges* from observable gluing the composite claim is [SPEC]. We restate this in §10.2 as a census table.

1.4 Contributions and outline

Concretely, this Part contributes the following.

1. A precise site Ctx of *measurement contexts* and a condensed instantiation of it built on Part I's pro-étale site (§3.1).
2. The definition of a *condensed sheaf of observables* $\mathcal{O} : \mathsf{Ctx}^{\mathrm{op}} \to \mathcal{V}$ and its dual *net* (precosheaf) of algebras $\mathfrak{A} : \mathsf{Ctx} \to \mathrm{Alg}(\mathcal{V})$, with an explicit treatment of the sheaf-versus-cosheaf fork that separates restriction-type from composition-type observables (§5, §4).
3. Two structural theorems, both [EST]: the *descent theorem* (Theorem 5.1), that pairwise-compatible local observations glue to a unique global observation; and the *automorphism-retention theorem* (Theorem 5.4), that gluing on the associated prestack preserves automorphism/gauge data, $\underline{\mathrm{Aut}}_{[X/G]}(x) \cong \mathrm{Stab}_G(x)$.
4. A precise account of *derived observables* (§6): the naive presheaf of values need not be a sheaf, and even a genuine sheaf of "locally trivial" observations can fail to globalize, the obstruction being a class in $\check{H}^1$ (Proposition 6.3); hence the observable one should record is the derived complex $R\Gamma(U, \mathcal{O})$.
5. The *reconstruction* theme (§7): Gelfand duality as the [EST] prototype of "geometry from observables", with Tomita–Takesaki, Connes, and Tannakian reconstruction as three further precedents, and the condensed generalization forwarded to Part V as [SPEC].
6. A precise formulation of the central diagram (1) and of the *Observable-First Principle* (§8), with each arrow's status made explicit.
7. Accompanying runnable Haskell (§9.2): a sheaf of observables with a descent-equalizer checker, a nonzero Čech obstruction to gluing, and a toy net with verified isotony and locality.

Section 2 fixes the categorical framework; Section 3 develops the functorial account of measurement; Section 4 reviews the established AQFT/factorization precedents; Section 5 defines condensed sheaves of observables and proves descent and automorphism retention; Section 6 treats derived observables; Section 7 treats reconstruction; Section 8 states the central principle; Section 9 collects results and the Haskell formalization; Section 10 discusses non-identifications, limitations, and what Part II hands downstream.

2 Mathematical Framework

We fix notation and recall exactly the sheaf-theoretic and condensed machinery we need. Nothing in this section is new; it is assembled so the paper is self-contained and so the status of each later step is unambiguous. All of §2 is [EST].

2.1 Sites, presheaves, sheaves, descent

Definition 2.1 (Site)**.** A *site* is a small category $\mathcal{C}$ equipped with a Grothendieck topology J: for each object U a collection $J(U)$ of *covering families* $\{f_i : U_i \to U\}_{i \in I}$, closed under base change (stability), composition (transitivity), and containing all isomorphisms (identity). We assume $\mathcal{C}$ has the fibre products $U_i \times_U U_j$ used below, or work with covering sieves so that they are not needed [16, 17, 19, 20].

Definition 2.2 (Presheaf, sheaf)**.** Let $\mathcal{V}$ be a category with small products and equalizers (e.g. Set, Ab, Cond(Ab)). A *$\mathcal{V}$-valued presheaf* on $\mathcal{C}$ is a functor $\mathcal{F} : \mathcal{C}^{\mathrm{op}} \to \mathcal{V}$. It is a *sheaf* if for every covering family $\{U_i \to U\} \in J(U)$ the diagram

$$\mathcal{F}(U) \xrightarrow{e} \prod_i \mathcal{F}(U_i) \underset{q}{\overset{p}{\rightrightarrows}} \prod_{i,j} \mathcal{F}(U_i \times_U U_j) \tag{2}$$

is an equalizer in $\mathcal{V}$, where p, q are induced by the two projections $U_i \times_U U_j \to U_i, U_j$. Explicitly (in Set): e sends $s \in \mathcal{F}(U)$ to the family of its restrictions $(s|_{U_i})_i$; and the equalizer condition says $\mathcal{F}(U)$ *is* the set of families $(s_i)_i \in \prod_i \mathcal{F}(U_i)$ that agree on overlaps, $s_i|_{U_i \times_U U_j} = s_j|_{U_i \times_U U_j}$. $\mathcal{F}$ is *separated* if e is a monomorphism (local data determine global data), and a full sheaf if additionally every compatible family descends (local data glue).

The equalizer (2) packages two independent physical demands into one categorical statement: *uniqueness* (a global observation is determined by its local restrictions) and *existence* (compatible local observations assemble into a global one). Separatedness is uniqueness alone; the sheaf condition adds existence.

Definition 2.3 (Sheafification)**.** The inclusion of sheaves into presheaves, $\mathrm{Sh}(\mathcal{C}, J) \hookrightarrow \mathrm{PSh}(\mathcal{C})$, admits a left adjoint $\mathcal{F} \mapsto \mathcal{F}^+$ (sheafification / associated sheaf), constructed by two applications of the plus-construction

$$\mathcal{F}^+(U) = \operatorname*{colim}_{\{U_i \to U\}} \check{H}^0(\{U_i \to U\}, \mathcal{F})$$

over the cofiltered system of covers. Sheafification is exact and preserves finite limits [19, 17].

2.2 The condensed site of Part I

Definition 2.4 (Condensed set; the site $*_{\text{pro-ét}}$)**.** Following Part I [1] and Scholze [2]: the *pro-étale site of a point* $*_{\text{pro-ét}}$ is the category ProFin of profinite sets with covers the finite jointly surjective families $\{S_i \to S\}$. A *condensed set* is a sheaf $T : \mathrm{ProFin}^{\mathrm{op}} \to \mathrm{Set}$ with $T(\varnothing) \cong *$, $T(S \sqcup S') \cong T(S) \times T(S')$, and the equalizer condition (2) for surjections $S' \twoheadrightarrow S$. Condensed abelian groups Cond(Ab) are the internal abelian group objects; they form a Grothendieck abelian category with enough injectives and projectives — the abelian-category repair that motivates the whole theory [2]. We write ExtDisc $\subset$ ProFin for the full subcategory of extremally disconnected (Stonean) profinite sets, which are the projective objects and form a basis for the site.

We will use two facts from Part I as a black box. (i) $\mathrm{Top}_{\mathrm{cg}} \hookrightarrow \mathrm{Cond(Set)}$ is fully faithful, so "a space" may be taken to mean "a condensed set". (ii) Sheaf theory depends only on the topology generated, so we may compute condensed sheaves on any of the equivalent generating sites (ProFin, ExtDisc, CompHaus).

2.3 Cosheaves and the limit/colimit duality

Definition 2.5 (Cosheaf)**.** A $\mathcal{V}$*-valued cosheaf* on $\mathcal{C}$ is a covariant functor $\mathfrak{A} : \mathcal{C} \to \mathcal{V}$ such that for every cover $\{U_i \to U\}$ the diagram

$$\coprod_{i,j} \mathfrak{A}(U_i \times_U U_j) \ \rightrightarrows\ \coprod_i \mathfrak{A}(U_i) \ \to\ \mathfrak{A}(U) \tag{3}$$

is a coequalizer (colimit) in $\mathcal{V}$: global sections are *assembled from* local ones by gluing along overlaps, dually to (2). Composition-type data (operators, algebras generated by local measurements) are naturally cosheaves. §4 shows this is exactly the sheaf/cosheaf fork separating a Haag–Kastler net (isotony = covariant) from a value functional (restriction = contravariant).

Remark 2.6 (The coproduct in Alg($\mathcal{V}$)). When $\mathcal{V} = \mathrm{Alg}(-)$ is a category of algebra objects, the coproduct $\coprod$ in (3) is *not* a disjoint union: it is the categorical coproduct of algebras, i.e. the (unital) free product, or the maximal tensor product for C^*-algebras. The colimit (3) therefore glues local algebras by *amalgamation over their overlaps*, which is stronger than the spatial "algebra generated inside a fixed representation" used in concrete AQFT. We return to this distinction — categorical codescent versus spatial additivity — in §5 (Remark 7.5), where it materially qualifies the specialization to a net.

2.4 Notational conventions

$\mathcal{V}$ denotes a fixed target category of "value objects," always one of Set, Ab, Cond(Set), Cond(Ab), or a category Alg($\mathcal{V}$) of (associative, unital) algebra objects therein. "Observable sheaf" means a $\mathcal{V}$-valued sheaf on the context site of §3.1; "net" means a Alg($\mathcal{V}$)-valued cosheaf/precosheaf. We write $R\Gamma$ for derived global sections, $\check{H}^i$ for Čech cohomology, Gpd for the 2-category of groupoids, and $\underline{G}$ for the sheaf represented by a group G (the condensed group G). We reserve $[X/G]$ for the quotient stack and $\mathrm{Stab}_G(x)$ for a point stabilizer.

3 Observables as Functors

We now build the observable layer in the operational order. The guiding idea is that a measurement is not a number attached to a point of a pre-given space, but a *functor*: a rule that respects the relations of refinement and compatibility among contexts. That a physical theory can *be* a functor is not new: an Atiyah–Segal topological quantum field theory is literally a symmetric monoidal functor from a cobordism category [23], rigidly reconstructible from its value on a point (the cobordism hypothesis [24]) — the first rigorous instance of "physics as a functor," and a template for the functorial account developed here. This section is definitional and, taken by itself, [EST] as mathematics; the claim that it is the *right* primitive for physics is [HEU], flagged as such.

3.1 Measurement contexts

Definition 3.1 (Measurement context; context site)**.** A *measurement context* is an operationally coherent family of jointly performable measurements — a "window" through which a system is probed. We model contexts as the objects of a small category Ctx equipped with a Grothendieck topology J_{Ctx}, subject to:

(C1) morphisms $V \to U$ are *refinements* ("V is a sub-context/sharper window of U", or a coarse-graining, depending on variance; we fix the contravariant convention below);

(C2) a covering family $\{U_i \to U\} \in J_{\mathsf{Ctx}}(U)$ is a way of *resolving* U into sub-contexts that jointly exhaust it, i.e. every measurement in U is determined by its restrictions to the U_i;

(C3) fibre products $U_i \times_U U_j$ exist and model the *joint* context in which the measurements common to U_i and U_j are performed.

We call $(\mathsf{Ctx}, J_{\mathsf{Ctx}})$ the *context site*.

Definition 3.1 is deliberately abstract: it is the interface, and different physical theories instantiate it differently. Three instantiations matter for us.

Example 3.2 (Geometric contexts: the AQFT instantiation)**.** Let $\mathsf{Ctx} = \mathcal{R}(M)$ be the poset of bounded, causally convex open regions of a globally hyperbolic spacetime M, with morphisms the inclusions and covers the families $\{R_i \hookrightarrow R\}$ with $\bigcup_i R_i = R$. This is the classical Haag–Kastler index category (§4). It is *geometry-first*: M is fixed in advance.

Example 3.3 (Algebraic contexts: the Isham–Döring-style instantiation)**.** Let $\mathsf{Ctx} = \mathcal{V}(\mathcal{N})$ be the poset of commutative (von Neumann) subalgebras of a fixed algebra of observables $\mathcal{N}$, ordered by inclusion. Here a "context" is a maximal family of simultaneously measurable quantities, exactly the "classical windows" of the topos approach to quantum theory [31, 32]. We flag at once (and expand in §10.1) that this is a *related but distinct* program: it reformulates quantum *logic* on a context category, whereas we aim at spacetime/observable structure. We borrow the context idea, not the program.

Example 3.4 (Condensed contexts: the instantiation of this Part)**.** Let $\mathsf{Ctx} \subseteq \mathrm{Cond}(\mathrm{Set})$ be a small full subcategory of condensed anima (Part I's spaces), containing the extremally disconnected profinite sets as "elementary probes," with J_{Ctx} the restriction of the pro-étale topology. A context is now a condensed *probe object* S, and a measurement in context S is a section of an observable sheaf over S. This is the *geometry-neutral* instantiation: no manifold is fixed; the "space" the contexts refer to will be reconstructed (§7), not assumed. All structural statements below are proved for a general context site and specialize to each of Examples 3.2 to 3.4.

3.2 The observable functor

Definition 3.5 (Observable presheaf; observable sheaf)**.** An *observable presheaf* (valued in $\mathcal{V}$) is a functor $\mathcal{O} : \mathsf{Ctx}^{\mathrm{op}} \to \mathcal{V}$. For a refinement $f : V \to U$ we write $\rho^U_V := \mathcal{O}(f) : \mathcal{O}(U) \to \mathcal{O}(V)$ (*restriction of an observation to a sharper context*). $\mathcal{O}$ is an *observable sheaf* if it satisfies descent (2) for every J_{Ctx}-cover. We call $\mathcal{O}(U)$ the *observations available in context* U and its elements *local observations*. When $\mathcal{V} = \mathrm{Cond}(\mathrm{Ab})$ or $\mathcal{V} = \mathrm{Cond}(\mathrm{Set})$ we speak of a *condensed sheaf of observables*.

Two features distinguish Definition 3.5 from the naive picture "an observable is a real-valued function on a state space." First, it is *functorial*: an observation is inseparable from its behaviour under refinement, which is what makes gluing meaningful. Second, its *values* live in a condensed category, so a "value" can itself carry topology, a group action, or — once we pass to the derived setting (§6) — higher homotopical data. This is the technical sense in which we "represent" rather than "coordinatize": the observable records a representation of the context, and only after reconstruction does that representation acquire coordinates.

3.3 Representation, not coordinates

The contrast with Part I is worth stating sharply, because it is the conceptual engine of the whole program.

Principle 3.6 (Representation over coordinates)**.** In the manifold picture, an observable is presented *in coordinates*: pick a chart $\varphi : U \to \mathbb{R}^n$, and a field is a function of φ. Coordinates are prior; the observable is their function. In the sheaf picture, an observable is presented *as a representation*: it is a compatible assignment $U \mapsto \mathcal{O}(U)$ respecting refinement, and *the coordinates, if any, are recovered afterwards* as the spectrum/realization of $\mathcal{O}$ (§7). Symbolically,

$$\underbrace{\text{coordinates} \;\to\; \text{observable}}_{\text{Part I, manifold grammar}} \qquad \text{is replaced by} \qquad \underbrace{\text{observable} \;\to\; \text{coordinates}}_{\text{Part II, sheaf grammar}}.$$

Principle 3.6 is not yet a theorem; it is the design commitment that the rest of the Part cashes out. Its established anchor is Gelfand duality (§7), which shows the right-hand replacement is literally correct for commutative C^*-algebras: the space is the spectrum of its own observable algebra. Its speculative content — that the same inversion is foundationally correct for *quantum* observables and *Lorentzian* geometry — is [SPEC] and forwarded to Part V.

4 The Established Precedent: Nets and Factorization Algebras

Before condensing anything, we record the two rigorous, established formalisms that already assign observables to regions and glue them: algebraic quantum field theory and factorization algebras. Both are [EST]. They are also where the *sheaf-versus-cosheaf fork* first appears, and we resolve it here so that later sections can speak unambiguously.

4.1 Haag–Kastler nets

Definition 4.1 (Haag–Kastler net)**.** A *Haag–Kastler net* on Minkowski space is an assignment $R \mapsto \mathfrak{A}(R)$ of unital C^*-algebras to bounded open regions, satisfying [4]:

- **Isotony:** $R_1 \subseteq R_2 \;\Rightarrow\; \mathfrak{A}(R_1) \subseteq \mathfrak{A}(R_2)$;
- **Einstein causality (microcausality):** if R_1, R_2 are spacelike separated then their algebras commute, $[\mathfrak{A}(R_1), \mathfrak{A}(R_2)] = 0$;
- **Covariance:** a representation of the Poincaré group acts compatibly, $\alpha_g(\mathfrak{A}(R)) = \mathfrak{A}(gR)$;
- (often) **Time-slice axiom:** the algebra of a Cauchy-surface neighbourhood already generates the whole net.

Isotony says $\mathfrak{A}$ is a covariant functor on the inclusion poset — the *net* is a precosheaf, not a presheaf. Its gluing, when the time-slice/additivity axioms hold, is by generation (colimit), matching (3).

Definition 4.2 (Locally covariant QFT, Brunetti–Fredenhagen–Verch)**.** A *locally covariant QFT* is a covariant functor $\mathfrak{A} : \mathsf{Loc} \to C^*\mathrm{Alg}$ from the category Loc of globally hyperbolic spacetimes (morphisms: isometric, causality-, orientation-, and time-orientation-preserving open embeddings) to unital C^*-algebras, satisfying isotony and Einstein causality *functorially over all spacetimes at once* [5]. This is the rigorous form of "background independence at the level of which spacetime one works on": $\mathfrak{A}$ is a functor whose *source* is a category of geometries. It is the closest established statement to what our context site abstracts.

Definition 4.2 already performs half of our inversion: it makes the assignment of observables a functor on a category of geometries, rather than a fixed geometry. We push the remaining half by (i) replacing Loc with the geometry-neutral context site of Example 3.4, and (ii) requiring descent, so that the functor is a *sheaf*, not merely a functor.

4.2 Factorization algebras

Definition 4.3 (Prefactorization algebra; factorization algebra)**.** A *prefactorization algebra* on M valued in cochain complexes is an assignment $U \mapsto \mathcal{F}(U) \in \mathrm{Ch}$ with, for pairwise disjoint $U_1, \ldots, U_n \subseteq V$, structure maps $\mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_n) \to \mathcal{F}(V)$, associative and equivariant. It is a *factorization algebra* if it satisfies the Weiss-cover gluing axiom: for every

Weiss cover $\mathfrak{U}$ of V, the natural map $\operatorname{colim}_{\mathfrak{U}} \check{C}(\mathfrak{U}, \mathcal{F}) \to \mathcal{F}(V)$ is a quasi-isomorphism [6]. Observables here *compose* (colimit gluing), not merely restrict.

Theorem 4.4 (Gwilliam–Rejzner comparison, [7])**.** [EST] *For free (Gaussian) field theories there is an explicit comparison functor relating the Costello–Gwilliam factorization algebra of observables to the locally covariant AQFT net: the factorization product corresponds to the algebraic product, and the two encodings of the same free theory determine each other.*

4.3 Resolving the sheaf-versus-cosheaf fork

Definitions 4.1 and 4.3 glue by colimits; Definition 3.5 glues by limits. This is not a contradiction but a duality, and both directions are physically real. We state the resolution as a principle we adhere to for the rest of the Part.

Principle 4.5 (Two kinds of observable data, and where quantum states break descent)**.** *Operator data* — the algebra *generated* by the measurements available in a context — extends along inclusions and assembles as a **precosheaf**/factorization algebra (colimit-type, (3)): a global algebra is generated by its local subalgebras. *Value data* splits into two sharply different cases. (a) *Classical* value data — a configuration of a sheaf of mutually commuting (Gelfand-dual) observables, or any deterministic outcome assignment — restricts along refinements and glues as a genuine **sheaf** (limit, (2)): compatible local configurations determine and assemble a unique global one. (b) *Quantum states* — expectation functionals ω on the net — form only a **presheaf** $\mathcal{O}_\omega$ that *fails the sheaf condition* (it fails to be separated and fails effective descent): a state restricted to local subalgebras is a family of reduced states (marginals), and marginals neither determine the global state (failure of separatedness) nor always admit a global positive extension (failure of gluing). That failure is not a defect of the formalism but its most important feature: it is precisely the mathematical signature of *entanglement* across the cover, and it is what forces the passage to *derived* observables (§6) and is handed to Part III as the object of study. A complete observable formalism therefore carries the net $\mathfrak{A}$ (precosheaf), the classical sheaf, *and* the state presheaf $\mathcal{O}_\omega$ whose descent obstruction is entanglement.

When we speak of "the condensed sheaf of observables" $\mathcal{O}$ in structural statements (Theorem 5.1, Section 6), we mean an object *defined* by the descent axiom — the classical/kinematic case (a), where descent holds by fiat — and we study, separately and honestly, the extent to which the physical state assignment (b) satisfies it. The descent theorem below is a statement about any $\mathcal{O}$ that *is* a sheaf; Proposition 5.6 records precisely how and why the quantum state presheaf is *not* one.

5 Sheaves of Measurements on the Condensed Site

We now prove the two structural theorems of the Part. Both are [EST]: they are the sheaf-theoretic facts, specialized to the context site, that make the operational order (1) well-posed. Theorem 5.1 is the "gluing local observations" theorem; Theorem 5.4 is the "gluing retains gauge data" theorem.

5.1 Descent: unique gluing of compatible observations

Theorem 5.1 (Descent for observable sheaves)**.** [EST] *Let $\mathcal{O} : \mathsf{Ctx}^{\mathrm{op}} \to \mathcal{V}$ be an observable sheaf (Definition 3.5), $\mathcal{V}$ a category with small products and equalizers, and let $\{U_i \to U\}_{i \in I} \in J_{\mathsf{Ctx}}(U)$ be a cover. Suppose given local observations $s_i \in \mathcal{O}(U_i)$ that are* pairwise compatible*,*

$$\rho^{U_i}_{U_i \times_U U_j}(s_i) \;=\; \rho^{U_j}_{U_i \times_U U_j}(s_j) \qquad \text{for all } i, j. \tag{4}$$

Then there is a unique *global observation $s \in \mathcal{O}(U)$ with $\rho^U_{U_i}(s) = s_i$ for all i.*

Proof. By Definition 2.2, the sheaf condition asserts that $e : \mathcal{O}(U) \to \prod_i \mathcal{O}(U_i)$ is the equalizer of the pair $p, q : \prod_i \mathcal{O}(U_i) \rightrightarrows \prod_{i,j} \mathcal{O}(U_i \times_U U_j)$ induced by the two projections. The compatibility hypothesis (4) says exactly that the family $(s_i)_i$, viewed as an element of $\prod_i \mathcal{O}(U_i)$, is equalized: $p\big((s_i)_i\big) = q\big((s_i)_i\big)$, since the (i,j) component of p is $\rho^{U_i}_{U_i \times_U U_j}(s_i)$ and of q is $\rho^{U_j}_{U_i \times_U U_j}(s_j)$. By the universal property of the equalizer, there is a unique $s \in \mathcal{O}(U)$ with $e(s) = (s_i)_i$, i.e. with $\rho^U_{U_i}(s) = s_i$ for all i. Existence is the factorization through the equalizer; uniqueness is the monomorphism property of e (separatedness). In $\mathcal{V} = \mathrm{Set}$ this is the elementary statement that $\mathcal{O}(U)$ is in bijection with the set of compatible families; in $\mathcal{V} = \mathrm{Cond(Ab)}$ the same universal property holds because Cond(Ab) is complete, so the equalizer exists and is computed pointwise on each extremally disconnected probe. □

Remark 5.2 (Why this needs the *sheaf*, not just a presheaf)*.* For a mere presheaf, (4) does not imply the existence of s; one obtains at most a formal family with no global representative. The content of $\mathcal{O}$ being a sheaf is precisely the promise that compatible local observations are *always* realized globally. Where that promise fails — where local data are compatible but do not glue — one is looking at a presheaf that is not a sheaf, and the failure is measured cohomologically. The physically decisive instance is the quantum state presheaf, which fails descent exactly because of entanglement (Proposition 5.6). That failure is not a defect to be legislated away; it is the source of *derived* observables (§6).

5.2 Automorphism retention: gluing does not forget gauge

The descent theorem, as stated, glues *values*. But physical observations frequently come with *symmetry*: a local observation is only defined up to a gauge transformation, and the gauge group acts on it. If we glued naively by passing to isomorphism classes — a coarse quotient — we would discard exactly the automorphism data that gauge theory (and, downstream, entanglement) depends on. The fix, inherited from the four-fold-proved theorem family of the seed projects [34], is to let observables be valued in *groupoids* (a prestack), so gluing is stack-theoretic and retains automorphisms.

Definition 5.3 (Observable prestack)**.** An *observable prestack* is a pseudofunctor $\mathfrak{R} : \mathsf{Ctx}^{\mathrm{op}} \to \mathsf{Gpd}$ assigning to each context a groupoid of observations-with-symmetry and to each refinement a restriction functor, coherently. It is an *observable stack* if it satisfies descent in the 2-categorical sense: groupoids of local observations-with-descent-data are equivalent to the groupoid of global observations [17, 18].

Theorem 5.4 (Automorphism retention)**.** [EST] *Let a symmetry group G act on an object X (a "local observable with G-gauge"), and let $[X/G]$ denote the quotient prestack $U \mapsto \{\, G\text{-equivariant maps } U \to X \,\}$, presented as a groupoid. Then for the object $x \in X$ regarded as a point of the stack $[X/G]$,*

$$\underline{\mathrm{Aut}}_{[X/G]}(x) \;\cong\; \mathrm{Stab}_G(x), \tag{5}$$

the automorphism group of x inside $[X/G]$ is the stabilizer of x in G. In particular, gluing on $[X/G]$ retains the full stabilizer (gauge) data, whereas the coarse set-quotient X/G retains only the orbit Gx and forgets $\mathrm{Stab}_G(x)$.

Proof. Objects of the groupoid $[X/G](*)$ over a point are elements $x \in X$; a morphism $x \to x'$ is a $g \in G$ with $gx = x'$. Hence $\mathrm{Hom}_{[X/G]}(x, x') = \{g \in G : gx = x'\}$, and in particular $\underline{\mathrm{Aut}}_{[X/G]}(x) = \mathrm{Hom}_{[X/G]}(x, x) = \{g \in G : gx = x\} = \mathrm{Stab}_G(x)$, which is (5). The isomorphism class $[x]$ in $\pi_0[X/G]$ is the orbit Gx, i.e. the point of the coarse quotient X/G; the coarse quotient therefore records π_0 only and discards the automorphism groups $\pi_1([X/G], x) = \mathrm{Stab}_G(x)$. That $[X/G]$ is a stack (satisfies descent) is the standard stackification statement for the action groupoid [17, 18]; descent then glues the automorphism data along with the objects. □

Remark 5.5 (Physical reading). [HEU] Theorem 5.4 is the precise sense in which "gluing local observations must retain automorphisms." An observable attached to a condensed probe is not a bare value but a value together with its gauge stabilizer; the correct global object is a stack, and the difference between the stack and its coarse quotient is exactly the gauge content that Parts III and VI will read as (respectively) protected quantum information and residual diffeomorphism/gauge redundancy. The one-level-down instance for condensed sheaves of observables is: an observable assigned to an extremally disconnected probe S must retain its automorphisms, not just its coarse value on S.

5.3 States form a presheaf, not a sheaf: the entanglement obstruction

It is essential to be honest about which observable data actually satisfy descent. The *algebra* side always assembles (as a precosheaf); the classical/kinematic value side is a sheaf by construction. But the *quantum state* side — the physically central case — is only a presheaf, and its failure to be a sheaf is exactly entanglement. We prove this precisely, because it is the hinge between Theorem 5.1 (which is conditional on being a sheaf) and §6 (derived observables).

Proposition 5.6 (The state presheaf and the entanglement obstruction)**.** [EST] *Let $\mathfrak{A}$ be a Haag–Kastler net (Definition 4.1) on $\mathcal{R}(M)$ and let ω be a state on the global algebra. Define the* state presheaf $\mathcal{O}_\omega(R) := \omega|_{\mathfrak{A}(R)}$*, the restriction (marginal, reduced state) of ω to the local algebra, with restriction along an inclusion $R' \subseteq R$ given by further restriction of the functional. Then:*

(i) $\mathcal{O}_\omega$ is a presheaf: restriction of a functional along an inclusion of subalgebras is strictly functorial.

(ii) $\mathcal{O}_\omega$ *is in general* not separated. *Take spacelike-separated* R_1, R_2 *for which* $\mathfrak{A}(R_1 \cup R_2)$ *receives the canonical map from the (maximal) tensor product* $\mathfrak{A}(R_1) \otimes \mathfrak{A}(R_2)$ *granted by Einstein causality (in AQFT under the split property one uses instead the minimal/spatial tensor product; either way the map is an inclusion when* C^**-independence holds, and it is automatically so in the finite-dimensional model below). An entangled global state* ω *and the product* $\omega|_{\mathfrak{A}(R_1)} \otimes \omega|_{\mathfrak{A}(R_2)}$ *of its marginals share* all *local restrictions on the cover* $\{R_1, R_2\}$ *yet differ globally: reduced states (marginals) do not determine the joint state (the quantum marginal problem [30]).*

(iii) $\mathcal{O}_\omega$ *is in general* not a sheaf. *A family of pairwise-compatible local states need not admit* any *globally positive extension; when it does, the extension is non-unique by (ii). Hence the equalizer condition (2) fails on both counts.*

The precise obstruction to descent of $\mathcal{O}_\omega$ *is the mathematical signature of* entanglement *across the cover.*

Proof. (i) If $R'' \subseteq R' \subseteq R$ then $(\omega|_{\mathfrak{A}(R)})|_{\mathfrak{A}(R')}|_{\mathfrak{A}(R'')} = \omega|_{\mathfrak{A}(R'')}$ by associativity of restriction, and identities restrict to identities, so $\mathcal{O}_\omega$ is a functor $\mathcal{R}(M)^{\mathrm{op}} \to \mathrm{Set}$. (ii) A linear functional is *not* multiplicative: knowing $\omega(a)$ for $a \in \mathfrak{A}(R_1)$ and $\omega(b)$ for $b \in \mathfrak{A}(R_2)$ gives no control over $\omega(ab)$ for the cross term $ab \in \mathfrak{A}(R_1 \cup R_2)$. Concretely, on $\mathfrak{A}(R_1) \otimes \mathfrak{A}(R_2) \cong M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$ the Bell state $\omega = \langle \Phi^+ | \cdot | \Phi^+ \rangle$ and the product of its marginals $\frac{1}{2}\mathrm{tr} \otimes \frac{1}{2}\mathrm{tr}$ have identical marginals (both maximally mixed on each factor) but differ on $a \otimes b$; both therefore map to the same family under e, so e is not monic. (iii) Existence of a global positive extension with prescribed marginals is the quantum marginal (representability) problem, whose solution set is a proper convex subset of all overlap-compatible families and is generally empty; when nonempty it is a positive-dimensional convex set (many global states, one family of marginals). Both failures are exactly the negation of the equalizer property (2). □

Remark 5.7 (The flaw is the physics). [HEU] Proposition 5.6 is the corrected replacement for a tempting but false claim (that additivity would make states a sheaf); a state is a functional, not a homomorphism, so its values on a generating family do not determine it. The correct reading turns the obstruction into the content: descent of the state presheaf holds iff there is no entanglement across the cover, so the *failure* of descent measures entanglement. This is why the honest observable attached to a context is not Γ but the derived complex $R\Gamma$ (§6), and it is the precise object Part III studies via condensed/derived cohomology. What glues cleanly is the net (algebra) and the classical sheaf; what refuses to glue — and thereby encodes global correlation — is the quantum state.

6 Derived Observables

The naive presheaf of observable values is, in general, *not* a sheaf, and even a genuine sheaf of "locally standard" observations can fail to admit a global observation. Both phenomena are cohomological. This section makes them precise and draws the moral: the observable one should attach to a context U is not the set of global sections $\Gamma(U, \mathcal{O}) = \mathcal{O}(U)$ but the *derived* complex $R\Gamma(U, \mathcal{O})$, whose higher cohomology records genuine, physically meaningful gluing obstructions. The mathematics of this section is [EST]; the reading of $H^{\geq 1}$ as "higher observable data" is [HEU], flagged where it occurs.

6.1 Why sheafification is forced

Proposition 6.1 (Presheaf quotients need not be sheaves)**.** [EST] *Let* $0 \to \mathcal{G} \to \mathcal{F} \to \mathcal{Q} \to 0$ *be a short exact sequence of abelian sheaves on a site (e.g. on the condensed site* $*_{\text{pro-ét}}$*). The naive presheaf quotient* $U \mapsto \mathcal{F}(U)/\mathcal{G}(U)$ *is separated but in general* not *a sheaf; its sheafification is* $\mathcal{Q}$*, and the comparison sits in the long exact cohomology sequence*

$$0 \to \mathcal{G}(U) \to \mathcal{F}(U) \to \mathcal{Q}(U) \xrightarrow{\delta} H^1(U, \mathcal{G}) \to H^1(U, \mathcal{F}) \to \cdots, \tag{6}$$

so the failure of a global section of $\mathcal{Q}$ *to lift to* $\mathcal{F}$ *is measured by the connecting map* δ *into the derived cohomology* $H^1(U, \mathcal{G})$*.*

Proof. Left-exactness of the section functor gives $0 \to \mathcal{G}(U) \to \mathcal{F}(U) \to \mathcal{Q}(U)$, so $\mathcal{F}(U)/\mathcal{G}(U)$ injects into $\mathcal{Q}(U)$; but this injection is generally not surjective, because a section of $\mathcal{Q}$ over U lifts to $\mathcal{F}$ only *locally* on some cover, and the local lifts differ on overlaps by a $\mathcal{G}$-valued cocycle whose class in $H^1(U, \mathcal{G})$ is δ of the section. That the sheafification of $U \mapsto \mathcal{F}(U)/\mathcal{G}(U)$ is $\mathcal{Q}$ is the definition of $\mathcal{Q}$ as the cokernel sheaf. The long exact sequence (6) is the standard cohomology sequence associated with the short exact sequence of sheaves [16, 19, 22]; for any *particular* cover $\mathfrak{U}$ the connecting map lands in $\check{H}^1(\mathfrak{U}, \mathcal{G})$, and passing to the colimit over covers (or to a Stonean hypercover, Proposition 6.7) recovers the derived $H^1(U, \mathcal{G})$. □

Reading Proposition 6.1 operationally: a "ratio" or "quotient" observable — one defined only modulo a subsheaf of gauge/reference degrees of freedom — is, at the presheaf level, *not* something local observations determine globally. One must sheafify, and the price of sheafifying is a genuine obstruction living in $\check{H}^1$.

6.2 Locally trivial observables and the $\check{H}^1$ obstruction

The cleanest, and physically most suggestive, source of derived observables is the *torsor*: an observable that is locally isomorphic to a fixed model $\underline{G}$ but globally twisted.

Definition 6.2 ($\underline{G}$-torsor observable)**.** Let G be a group and $\underline{G}$ its sheaf on Ctx. A $\underline{G}$*-torsor* (principal homogeneous sheaf) is a sheaf P with a free, transitive $\underline{G}$-action that is *locally trivial*: there is a cover $\{U_i \to U\}$ with $P|_{U_i} \cong \underline{G}|_{U_i}$ equivariantly. Physically: a system of measurements each of which looks, in every sufficiently fine context, like a standard G-labelled apparatus, but whose labellings need not agree globally.

Proposition 6.3 (Classification of torsor observables)**.** [EST] *Isomorphism classes of* $\underline{G}$*-torsors on* U *are in natural bijection with the pointed nonabelian cohomology* $\check{H}^1(\mathsf{Ctx}/U, \underline{G})$*. A torsor* P *admits a global section (is trivial, i.e. globally isomorphic to* $\underline{G}$*) if and only if its class* $[P] \in \check{H}^1$ *vanishes. In particular, pairwise-compatible local trivializations* $P|_{U_i} \cong \underline{G}|_{U_i}$ *need* not *assemble into a global trivialization; the obstruction is precisely* $[P]$*.*

Proof. Choosing local trivializations $\varphi_i : P|_{U_i} \xrightarrow{\sim} \underline{G}|_{U_i}$, the comparisons $g_{ij} := \varphi_i\varphi_j^{-1}$ on $U_i \times_U U_j$ are $\underline{G}$-valued and satisfy the cocycle identity $g_{ij}g_{jk} = g_{ik}$ on triple overlaps; a change of trivializations $\varphi_i \mapsto h_i\varphi_i$ replaces g_{ij} by $h_i g_{ij} h_j^{-1}$, so the class $[g_{ij}] \in \check{H}^1(\mathsf{Ctx}/U, \underline{G})$ is well defined and depends only on P. Conversely a cocycle glues $\coprod_i \underline{G}|_{U_i}$ into a torsor, and the two constructions are mutually inverse on isomorphism classes. A global trivialization is a 0-cochain (h_i) with $g_{ij} = h_i^{-1}h_j$, i.e. a coboundary, so it exists iff $[P] = 0$. This is Giraud's nonabelian H^1 [18, 16]. □

Example 6.4 (A nonzero obstruction: monodromy)**.** Take $G = \mathbb{Z}/2$ and a "loop" context covered by two arcs meeting in two components, with transition data $+1$ on one overlap component and -1 on the other. The cocycle is nontrivial: its class in $\check{H}^1 \cong \mathbb{Z}/2$ is the nonzero element, so the $\mathbb{Z}/2$-torsor has no global section, even though it is trivial on each arc. This is the sheaf-theoretic shadow of monodromy (a Möbius-type twist), and it is the smallest genuinely *derived* observable: local observations exist and are pairwise compatible up to the cocycle, yet no global observation exists. The Haskell of §9.2 computes this class explicitly as the product of transition signs around the loop.

6.3 The derived observable complex

Propositions 6.1 and 6.3 both say the same thing: $H^0 = \Gamma$ is not enough. The systematic remedy is to replace the section functor by its right-derived functor.

Definition 6.5 (Derived observables)**.** For an observable sheaf $\mathcal{O}$ valued in an abelian (or stable) condensed category, the *derived observables* of a context U are the object $R\Gamma(U, \mathcal{O}) \in D(\mathcal{V})$ of the derived ∞-category, with cohomology the *higher observables* $H^i(U, \mathcal{O}) := H^i R\Gamma(U, \mathcal{O})$. Degree 0 recovers the ordinary global observations $H^0(U, \mathcal{O}) = \mathcal{O}(U)$; degrees $i \geq 1$ record the successive gluing obstructions. For any cover $\{U_i \to U\}$ there is a Čech-to-derived (descent) spectral sequence

$$E_2^{p,q} = \check{H}^p\big(\{U_i \to U\},\, \underline{H}^q(\mathcal{O})\big) \implies H^{p+q}(U, \mathcal{O}), \tag{7}$$

with coefficient the *presheaf* $\underline{H}^q(\mathcal{O}) : V \mapsto H^q(V, \mathcal{O})$. In general it does not collapse; Proposition 6.7 explains when, and via which resolutions, it computes $R\Gamma$ on the condensed site.

Remark 6.6 (Why "derived" is not optional). [HEU] Three independent forces make the derived complex the correct observable. (i) *Algebra*: quotient/ratio observables are only sheaves after sheafification, and sheafification manufactures H^1 (Proposition 6.1). (ii) *Topology*: locally trivial observables carry a genuine $\check{H}^1$ obstruction to globalization (Proposition 6.3). (iii) *Homotopy*: the values themselves may be condensed *anima* with internal higher structure, so "agree on overlaps" must be weakened to "agree up to coherent homotopy," which is descent in a stable ∞-category [21, 22] and is computed by $R\Gamma$, not Γ. In every case the naive presheaf undercounts; the derived complex is the honest bookkeeping of how local measurements fail — and, by their failure, encode global structure. This is the precise technical content of the slogan "derived observables see the global shape of the relations among measurements," and it is the bridge to Part III, where the classes $H^{\geq 1}$ are read as (protected) quantum information.

Proposition 6.7 (Derived observables via Stonean hypercovers)**.** [EST] *Let $\mathcal{O}$ be a sheaf valued in* Cond(Ab) *(or a stable condensed category), and let $S_\bullet \to U$ be a* hypercover by extremally disconnected probes*: an augmented simplicial object whose every level S_n is Stonean and whose matching maps $S_n \to (\operatorname{cosk}_{n-1} S_\bullet)_n$ are covers. Then the derived observables are the totalization of $\mathcal{O}$ evaluated levelwise,*

$$R\Gamma(U, \mathcal{O}) \;\simeq\; \operatorname{Tot}\big[\,\mathcal{O}(S_0) \to \mathcal{O}(S_1) \to \mathcal{O}(S_2) \to \cdots\,\big], \qquad H^p(U, \mathcal{O}) \cong H^p\big(\mathcal{O}(S_\bullet)\big), \tag{8}$$

the cohomology of the normalized cochain complex of the cosimplicial abelian group $\mathcal{O}(S_\bullet)$.

Proof. Each level S_n is extremally disconnected, hence $\mathcal{O}$-acyclic: $H^q(S_n, \mathcal{O}) = 0$ for $q > 0$, because the Stonean objects are the projective objects of the pro-étale site [2]. The descent (Bousfield–Kan) spectral sequence of the cosimplicial object $\mathcal{O}(S_\bullet)$ therefore has $E_1^{p,q} = H^q(S_p, \mathcal{O})$, which vanishes for $q > 0$ and equals $\mathcal{O}(S_p)$ for $q = 0$; it collapses onto the single row $q = 0$, whose totalization is the normalized complex of $\mathcal{O}(S_\bullet)$ and converges to $H^p(U, \mathcal{O})$. Equivalently, $S_\bullet$ is a resolution of U by projective (acyclic) objects, so evaluating $\mathcal{O}$ on it computes the right-derived functor $R\Gamma$, exactly as an acyclic resolution computes derived sections [16, 22]. □

Remark 6.8 (Why a hypercover, not a Čech nerve). [EST] It is essential that the resolution be a hypercover, with *every* level chosen Stonean, and *not* the Čech nerve of a single Stonean cover $\{S_i \to U\}$. The fibre products $S_i \times_U S_j$ appearing in a Čech nerve are in general *not* extremally disconnected — a product of infinite Stonean spaces is typically not Stonean — so the higher Čech terms are not $\mathcal{O}$-acyclic and $\underline{H}^q(\mathcal{O})$ need not vanish on them; the Čech-to-derived spectral sequence (7) then does not collapse and Čech cohomology can differ from derived cohomology. Refining each simplicial level to a Stonean object (which is always possible, since the Stonean objects form a basis) is exactly the repair, and is why derived observables on the condensed site are computed by projective simplicial resolutions rather than by the nerve of one cover.

7 Reconstruction of Global Structure

We reach the third arrow of (1): from the sheaf/cosheaf of observables and its descent data — the *relations* — back to a *geometry*. This is where the program's ambition is concentrated, and where the epistemic discipline earns its keep. The reconstruction *step* has impeccable established precedent in the commutative/classical case; its extension to the quantum, Lorentzian, condensed case is exactly the [SPEC] content forwarded to Part V. We keep the two apart.

7.1 The established prototype: Gelfand duality

Theorem 7.1 (Gelfand–Naimark duality, [8])**.** [EST] *The functor $X \mapsto C(X)$ from compact Hausdorff spaces to commutative unital C^*-algebras is a contravariant equivalence onto its image, with quasi-inverse $A \mapsto \operatorname{Spec}(A) := \operatorname{Hom}_{C^*\mathrm{Alg}}(A, \mathbb{C})$ (the Gelfand spectrum, with the* *weak-∗ topology). Consequently, a compact Hausdorff space is* recovered*, functorially and without loss, from its commutative algebra of continuous observables:*

$$X \;\cong\; \mathrm{Spec}\big(C(X)\big). \tag{9}$$

Theorem 7.1 is the rigorous kernel of the whole program's third arrow. It is a theorem, not a slogan: the space is *literally* the spectrum of its observable algebra, so "geometry from observables" is, in the commutative case, established mathematics. Two features generalize into our setting. First, it is contravariant — the sheaf/cosheaf duality of Principle 4.5 is its shadow. Second, and crucially for Part I compatibility: CompHaus is one of the generating sites for condensed sets (Definition 2.4), so (9) already lives natively in the condensed world.

Corollary 7.2 (Reconstruction on the condensed site, commutative case)**.** [EST] *A condensed commutative C^*-algebra object that is representable by $C(X)$ for $X \in$* CompHaus *determines X as its condensed spectrum. Hence, for commutative observable data, the context site's underlying space is reconstructed inside* Cond(Set) *by the same spectral functor.*

Proof. Immediate from Theorem 7.1 and the fully faithful embedding CompHaus $\hookrightarrow$ Cond(Set) of Part I [1, 2]: applying Spec to the algebra and viewing the result through the embedding lands the reconstructed space in Cond(Set). □

7.2 Three further precedents, and the boundary they mark

Beyond the commutative case, three established theorems recover geometric/dynamical data from algebraic/representation data, and together they mark exactly the boundary of what is proved.

- **Tomita–Takesaki and Bisognano–Wichmann** [EST] [9, 10, 11]: a von Neumann algebra with a cyclic separating vector determines a modular flow σ_t, and for a Rindler wedge in vacuum this flow *is* the boost, with modular conjugation encoding the causal complement. Geometric data (boosts, causal complements) are recovered from an algebra plus a state.

- **Connes reconstruction** [EST] [12]: a commutative spectral triple satisfying Connes's axioms is the canonical spectral triple of a closed Riemannian spin manifold — a manifold recovered from purely spectral/algebraic data, *with no manifold in the input.*

- **Tannakian reconstruction** [EST] [14, 15]: a neutral Tannakian category with a fibre functor ω reconstructs an affine group scheme G_ω with $\mathcal{C} \simeq \mathrm{Rep}_k(G_\omega)$ — a geometric object recovered from categorical representation data.

Each is a rigorous instance of "geometric/dynamical structure is an invariant of representation data." And each marks a boundary. Connes's theorem is *Riemannian* and *static*; a Lorentzian, dynamical reconstruction recovering causal (not just metric) structure is open [13]. Tannakian reconstruction targets $\mathrm{Vect}_k^{\mathrm{fd}}$, not a condensed target. The program's Part V will propose replacing these targets by condensed/solid/liquid ones and asking for an emergent *Lorentzian* geometry — and that proposal is [SPEC], not a theorem. We state this cleanly to prevent silent upgrading.

Principle 7.3 (Reconstruction status boundary)**.** [EST] for: reconstruction of a compact Hausdorff space from its commutative C^*-algebra (Gelfand); of a Riemannian spin manifold from a commutative spectral triple (Connes); of boosts and causal complements from an algebra with a cyclic separating state (Bisognano–Wichmann); of an affine group scheme from a neutral Tannakian category (Tannaka). [SPEC] for: reconstruction of a *Lorentzian, dynamical* geometry from a *condensed, quantum* sheaf of observables. Part II supplies the observable/relational input to the latter; it does not claim the output.

7.3 The specialization problem: recovering a net

Finally we make good on the Part's own theorem-shaped open question (KB §6.2): does a condensed sheaf of observables specialize to an ordinary Haag–Kastler net? We can prove the discrete/kinematic direction cleanly and flag the full statement as open.

Proposition 7.4 (Discrete specialization: a precosheaf is the kinematics of a net)**.** [EST] *Let* $\mathsf{Ctx} = \mathcal{R}(M)$ *be the region poset of Example 3.2, and let* $\mathfrak{A}$ *be a* Alg*-valued* precosheaf *on* Ctx *(a covariant functor) whose values are* discrete *C^*-algebras (objects of C^**Alg *regarded as condensed algebras via the discrete embedding), such that each structural map* $\mathfrak{A}(R_1) \to \mathfrak{A}(R_2)$ *for* $R_1 \subseteq R_2$ *is a monomorphism, and such that the images of* $\mathfrak{A}(R_1), \mathfrak{A}(R_2)$ *in* $\mathfrak{A}(R)$ *commute whenever* $R_1, R_2 \subseteq R$ *are spacelike separated. Then* $\mathfrak{A}$ *is exactly the data of a Haag–Kastler net satisfying isotony and Einstein causality (Definition 4.1); adding covariance is adding an isometry-equivariance of the functor. Conversely every Haag–Kastler net is such a precosheaf.*

Proof. A covariant functor on the inclusion poset with monic structural maps is, verbatim, an isotonous assignment $R \mapsto \mathfrak{A}(R)$ with $\mathfrak{A}(R_1) \subseteq \mathfrak{A}(R_2)$ for $R_1 \subseteq R_2$; the commutativity hypothesis for spacelike-separated regions is Einstein causality verbatim; equivariance under the isometry action is Poincaré covariance. Thus $\mathfrak{A}$ satisfies Definition 4.1, and every net gives such a precosheaf by reading isotony as functoriality. The discreteness of the target is the specialization: dropping it (allowing genuinely condensed, non-discrete C^*-algebra objects) is the "extra room" where non-net phenomena can live. □

Remark 7.5 (Spatial additivity is *not* categorical codescent). [EST] We deliberately state Proposition 7.4 for a *precosheaf*, not a cosheaf, because a spatial Haag–Kastler net generally *fails* the full codescent axiom (3). The categorical colimit in (3) for C^*-algebras is the amalgamated free product (or maximal tensor product), whereas the spatial additivity axiom defines $\mathfrak{A}(R_1 \cup R_2)$ as the C^*-subalgebra *spatially generated* inside a fixed ambient representation on $B(\mathcal{H})$. The spatial algebra is a specific *quotient* of the categorical colimit, imposing dynamical relations that amalgamation alone does not; so an additive net is a precosheaf whose spatial gluing is weaker than categorical codescent (cf. Remark 2.6). This is why we claim only the isotony/causality equivalence here, and do *not* claim that nets are cosheaves in the strict colimit sense.

Remark 7.6 (What remains open). [SPEC] Proposition 7.4 recovers the *kinematic* net (isotony and causality) from discrete-valued condensed data, with Remark 7.5 isolating exactly which gluing property does *not* come for free. The full conjecture — that a genuinely condensed (non-discrete) sheaf of observables on the geometry-neutral site of

Example 3.4, together with a reconstruction functor of §7, yields a locally covariant QFT *and* its underlying spacetime — is open, and is the joint deliverable of Parts IV–VI. We claim only the discrete kinematic specialization here.

8 The Central Idea: Measurement → Relations → Geometry

We now assemble the pieces into the Part's central formal statement. The three arrows of (1) have all appeared: measurement $\mapsto$ observable functor (§3); observable functor $\mapsto$ relational descent data (§5, §6); relational data $\mapsto$ geometry (§7). We make the pipeline a diagram and a principle, with each arrow's status explicit.

8.1 The pipeline diagram

$$\text{Measurements} \xrightarrow{\mathcal{O}} \begin{array}{c}\text{Condensed sheaf}\\ \text{of observables}\end{array} \xrightarrow{\text{descent}} \begin{array}{c}\text{Relational structure}\\ (\text{site} + R\Gamma\text{ data})\end{array} \xrightarrow{\text{Spec/Real}} \text{Geometry} \tag{10}$$

[EST] [EST] [HEU]/[SPEC]

Reading (10) left to right: a family of measurements is organized into an observable functor $\mathcal{O}$ (arrow 1, [EST] as a definition); the observable functor's descent/derived data — which compatible local observations glue, and the higher obstructions $H^{\geq 1}$ when they do not — constitute a purely *relational* structure, a site together with the complex $R\Gamma$ (arrow 2, [EST] by Theorem 5.1, Proposition 6.3, Definition 6.5); and a reconstruction/realization functor Spec/Real turns that relational structure into a geometry (arrow 3, [EST] in the commutative/Gelfand case, [HEU]/[SPEC] in the quantum, Lorentzian, condensed case, by Principle 7.3).

8.2 The Observable-First Principle

Principle 8.1 (Observable-First Principle)**.** Physical content is carried, *prior to any geometric structure*, by a condensed sheaf of observables $\mathcal{O}$ (with its dual net $\mathfrak{A}$) on a site of measurement contexts. The relations among measurements are the descent/derived data of $\mathcal{O}$; geometry is not assumed but is the image of $\mathcal{O}$ under a reconstruction functor Spec/Real. Formally, the arena of physics is taken to be the pair $(\mathsf{Ctx}, \mathcal{O})$, and any geometry $\mathcal{G}$ is defined by $\mathcal{G} := \mathrm{Real}\big(R\Gamma(-, \mathcal{O})\big)$, when such a Real exists.

Remark 8.2 (Status of the Principle). The *formalism* of Principle 8.1 — that one *may* package physics as $(\mathsf{Ctx}, \mathcal{O})$ and *define* a candidate geometry as a realization of $R\Gamma$ — is [EST]: it is a definition, and each ingredient exists. The *claim* that this is the *fundamental* order of physics, that real spacetime *is* such a reconstruction, is [SPEC]: it asserts the existence and physical correctness of a reconstruction functor Real in a regime (quantum, Lorentzian, condensed) where none is proved to exist. By the composition discipline (§1.3) any claim using arrow 3 of (10) inherits [SPEC], regardless of how rigorous arrows 1–2 are.

8.3 Lineage: ER = EPR, RT, and entanglement-wedge reconstruction

The arrow "relations → geometry" is not conjured from nothing; it is the AdS/CFT slogan "spacetime is built from entanglement" [29, 25, 28], lifted out of holographic duality and asked to hold generally. We flag the lineage precisely. The Ryu–Takayanagi formula and entanglement-wedge reconstruction [25, 26, 27] show, *within AdS/CFT and its toy models*, that a bulk region and its geometry are recovered from a boundary subalgebra of observables — a literal instance of "measurement reconstructs geometry." ER = EPR [28] proposes that geometric connectivity *is* entanglement. These are, in their native setting, at best [HEU]/[SPEC] (AdS/CFT is itself conjectural), and our arrow 3 inherits at most that status when read as a general mechanism. What Part II adds is a candidate *mathematical home* for the slogan — descent and derived sections of a condensed sheaf — so that "entanglement as gluing compatibility" can be made precise in Part III rather than remaining a metaphor. We claim the home, not the physics.

9 Results and Formalization

9.1 Summary of results

We collect the Part's deliverables and their status.

1. Theorem 5.1 ([EST]): compatible local observations glue uniquely — the operational content of the sheaf condition on the context site.
2. Theorem 5.4 ([EST]): gluing on the observable prestack retains automorphism/gauge data, $\underline{\mathrm{Aut}}_{[X/G]}(x) \cong \mathrm{Stab}_G(x)$.
3. Proposition 5.6 ([EST]): the quantum state presheaf is *not* a sheaf — it fails both separatedness and gluing — and the obstruction is exactly entanglement across the cover (the quantum marginal problem); this is the physically decisive reason observables must be derived.
4. Proposition 6.1, Proposition 6.3 ([EST]): the naive presheaf of observable values must be sheafified and derived; the first genuinely derived observable is a $\check{H}^1$-obstruction to gluing locally trivial observations, with an explicit nonzero example (Example 6.4).
5. Proposition 6.7 with Remark 6.8 ([EST]): on the condensed site, derived observables are computed by evaluating on a *hypercover* (projective simplicial resolution) by extremally disconnected probes — not by the Čech nerve of a single cover, whose fibre products are generally not Stonean.
6. Theorem 7.1, Corollary 7.2 ([EST]): Gelfand duality reconstructs a space from its commutative observable algebra, natively inside Cond(Set).
7. Proposition 7.4 with Remark 7.5 ([EST]): a discrete-valued precosheaf on the region poset is exactly the isotony/causality kinematics of a Haag–Kastler net; the discrete

hypothesis is the specialization, spatial additivity is weaker than categorical codescent, and dropping discreteness is the "extra room."

8. Principle 8.1 (formalism [EST]; foundational claim [SPEC]): the Observable-First Principle and the pipeline (10).

9.2 Haskell formalization

To keep the formal claims executable rather than merely asserted, the paper ships a small Haskell library (compiled warning-free under `-Wall` with GHC 9.14).[1] It is a finite, decidable toy of the constructions above; it proves nothing about the infinite condensed objects, but it makes the finite combinatorial heart of each theorem runnable and testable. Three modules and a driver:

- `Context.hs`: measurement contexts as finite index sets (Definition 3.1), with covers, pairwise overlaps (the nerve up to degree 2), and the refinement relation.
- `Observable.hs`: an observable (pre)sheaf as an assignment of sections to contexts (Definition 3.5); `restrict`; `compatible`; the descent-equalizer check `glue` realizing Theorem 5.1; and `cechClass`, which computes the $\mathbb{Z}/2$ Čech obstruction of Example 6.4 as the product of transition signs around a loop — a nonzero derived observable.
- `Net.hs`: a toy net (precosheaf) of algebras (Definition 4.1) with local generators; `isotone` verifies isotony (support inclusion under region inclusion), and `commute` verifies Einstein causality (elements with disjoint support commute, computed not asserted, because different sites commute by construction while same-site generators do not).
- `Main.hs`: runs all demonstrations, printing the glued global section of a compatible family, the failure to glue an incompatible family, the nonzero monodromy class, and the isotony/locality checks on the toy net.

The library is the computational witness that Theorem 5.1 (gluing), Proposition 6.3/Example 6.4 (the $\check{H}^1$ obstruction), and Definition 4.1 (isotony + locality) are internally consistent and correctly stated on their finite models.

10 Discussion

10.1 Explicit non-identifications

We follow the seed projects' discipline of naming, and refusing to silently merge, neighbouring programs.

[1]The library is provided as ancillary files with this submission, under `src/part2-observables-as-condensed-sheaves/`: the modules `Context.hs`, `Observable.hs`, `Net.hs`, and `Main.hs`. Run `runghc Main.hs`, or compile with `ghc -Wall Main.hs`.

- **Topos quantum theory (Isham–Döring)** [31, 32] uses presheaves on the context category $\mathcal{V}(\mathcal{N})$ of commutative subalgebras (Example 3.3) to reformulate quantum *logic* and the Kochen–Specker obstruction. We borrow the *context site* idea but aim at spacetime/observable structure, not quantum logic, and our target is condensed sheaves on the pro-étale site, not Set-valued presheaves on $\mathcal{V}(\mathcal{N})$. The programs share technology and differ in object.
- **Cohesive HoTT (Schreiber)** [33] is a *synthetic*, internal-language route to smooth ∞-groupoids via shape/flat/sharp modalities. Condensed mathematics is an *analytic*, external route via sheaves on profinite sets. They aim near the same targets by different means; we do not conflate "sheaf-theoretic physics" into one undifferentiated idea.
- **Factorization algebras vs. our sheaf** (§4.3): the factorization algebra is a cosheaf (colimit gluing of *operators*); our $\mathcal{O}$ is a sheaf (limit gluing of *values*). Principle 4.5 keeps both; Theorem 4.4 (Gwilliam–Rejzner) is the established bridge in the free case. We do not silently pick one side of the fork.
- **Gelfand reconstruction vs. the general claim**: Gelfand (Theorem 7.1) is a theorem for *commutative* algebras and *compact Hausdorff* spaces; the quantum/Lorentzian/ condensed reconstruction is not, and Principle 7.3 keeps them apart.

10.2 Limitations and status composition

The mathematical spine of this Part is entirely [EST]: sheaves, descent, Čech/derived cohomology, Gelfand duality, the AQFT and factorization-algebra formalisms. The *physical* thesis — that observables are condensed-sheaf-valued *prior* to geometry, and that geometry is their reconstruction — is [HEU]/[SPEC], and by the monotone, worst-component-wins composition rule (§1.3) any claim that reaches arrow 3 of (10) is [SPEC]. We tabulate the census, in the style the seed projects mandate.

Claim	Basis	Status
Sheaf descent: compatible local observations glue uniquely	Equalizer/sheaf condition (Theorem 5.1)	[EST]
Gluing retains gauge data, $\underline{\mathrm{Aut}}_{[X/G]} \cong \mathrm{Stab}_G$	Action groupoid / stack (Theorem 5.4)	[EST]
Quantum states are a presheaf, not a sheaf; the gap is entanglement	Marginal problem (Proposition 5.6)	[EST]
Derived observables: $\check{H}^1$ obstruction to gluing	Torsors, snake lemma (Proposition 6.3)	[EST]
Space = Spec of its observable algebra (commutative)	Gelfand–Naimark (Theorem 7.1)	[EST]
Discrete precosheaf = isotony/ causality of a Haag–Kastler net	Functoriality + locality (Proposition 7.4)	[EST]
Observables are prior to geometry	Observable-First formalism (Principle 8.1)	[HEU]
Geometry is reconstruction of a *condensed, Lorentzian* sheaf	arrow 3, general case (Principle 7.3)	[SPEC]

The honest headline: Part II proves the observable/relational layer rigorously and offers geometry only as a labelled hypothesis about the last arrow. The extra room noted in Proposition 7.4 and Remark 7.6 is where the hypothesis will be tested, not confirmed, downstream.

10.3 What Part II hands to Parts III–VII

The modular hand-offs are explicit.

- **To Part III (Information):** the derived observables of §6 — the classes $H^{\geq 1}(U, \mathcal{O})$ and the torsor obstruction of Proposition 6.3 — are the candidate mathematical home for "entanglement as sheaf/gluing compatibility." Part III replaces the target of $\mathcal{O}$ by condensed/solid vector spaces and reads the cocycles as (protected) quantum information.

- **To Part IV (Fields):** the net/precosheaf $\mathfrak{A}$ and Principle 4.5 are the target formalism to reproduce with fields valued in solid/liquid modules; the discrete specialization Proposition 7.4 is the base case to deform away from.

- **To Part V (Realization):** arrow 3 of (10), with the four precedents of §7 as the established scaffold and the condensed-Tannakian reconstruction as the open target; Part V is where Real is to be constructed or shown impossible.

- **To Parts VI–VII (Synthesis, Capstone):** the pipeline (10) and its status table are the exact objects the closure theorem of Part VI must show assemble into a stack, and the exact arrows Part VII's boxed hierarchy must exhibit — with the same, unmovable admission that only the last arrow is speculative.

11 Conclusion

Part I gave us condensed *spaces*; Part II has built the condensed *observable* layer on top of them and, in doing so, has inverted the usual order of physics. We defined a site of measurement contexts and a condensed sheaf of observables on it; we proved that compatible local observations glue uniquely (Theorem 5.1) and that the gluing retains gauge/automorphism data rather than collapsing to a coarse quotient (Theorem 5.4); we showed why the naive presheaf of values must be sheafified and derived, with an explicit nonzero $\check{H}^1$ obstruction as the first derived observable (Proposition 6.3, Example 6.4); and we exhibited Gelfand duality as the rigorous prototype of "geometry from observables," with the quantum, Lorentzian, condensed reconstruction cleanly labelled [SPEC] and forwarded to Part V. The central formal statement, the pipeline measurement $\to$ relations $\to$ geometry of (10) and the Observable-First Principle 8.1, is offered with its arrows' epistemic statuses on their sleeves: the first two established, the third a hypothesis.

The modular payoff is that everything downstream now has a precise object to act on. Entanglement (Part III) has the derived classes $H^{\geq 1}(U, \mathcal{O})$ to be read as compatibility data; fields (Part IV) have the net/precosheaf to reproduce condensed-analytically; geometry (Part V) has arrow 3 and its four precedents to try to make into a functor; and the synthesis (Parts VI–VII) has the whole pipeline to try to close into a stack. What Part II asserts, it proves; what it hopes, it labels. That discipline — observables before geometry, established before speculative — is the method the rest of the program will live or die by.

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