Information as the Primitive Object: Condensed Cohomology and Quantum Information

Part III of *Toward a Condensed Representation Theory of Physics*

Matthew Long
The YonedaAI Collaboration
YonedaAI Research Collective
Chicago, IL
matthew@yonedaai.com · https://yonedaai.com

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Abstract

Part II of this series established that the assignment of quantum *observables* to regions organizes into a condensed sheaf, but that the assignment of quantum *states* does not: the state functor is only a presheaf, and its failure to satisfy descent is exactly the phenomenon of entanglement, controlled by the quantum marginal problem. This paper takes that failure as its starting point and its subject. We argue that the natural measure of the obstruction to gluing local informational data is *cohomology*, and that the natural home for the obstruction classes is the derived category of condensed abelian groups, respectively condensed (solid/liquid) vector spaces. We separate three registers with scrupulous care. *Established mathematics* [EST]: Čech and derived-functor cohomology as the obstruction theory of gluing; the derived ∞-category $\mathcal{D}(\mathrm{Cond(Ab)})$ and $\mathcal{D}(\mathbb{Z}_{\mathrm{solid}})$; the surface-code theorem identifying logical operators with $H_1(\Sigma; \mathbb{Z}/2)$; the quantum marginal problem, monogamy, and strong subadditivity; and holographic quantum error correction (the HaPPY code, complementary recovery, the Ryu–Takayanagi formula) as rigorous toy models of state reconstruction. *Heuristic dictionary* [HEU]: the reading of a nonvanishing cohomology class as "protected" or "non-locally stored" quantum information. *Speculative hypothesis* [SPEC], clearly boxed and labelled: that entanglement *is* the non-reconstructibility of a system of condensed local states — a class in a condensed cosheaf (factorization) homology theory on unions, dual to the marginal-problem sheaf obstruction on overlaps — and that geometry emerges as the moduli of compatibility conditions among such informational structures. We flag prominently that no precedent in the literature combines condensed cohomology with entanglement; the central proposal is offered as a precise conjecture, contrasted throughout with what is genuinely established, and its composite epistemic status is honestly computed as [SPEC] under the worst-component-wins discipline of the series.

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1 Introduction

1.1 The modular program and its governing perspective

This is Part III of a modular, hierarchical seven-part program built on a single organizing perspective:

> *Physics is the study of realizations of condensed mathematical structures.*

The program is deliberately *modular*, not unified: each Part isolates one law of the translation between condensed mathematics and physics, states precisely what it establishes, and composes with its predecessors only through explicitly declared interfaces. Part I [1] replaced the smooth manifold by the condensed set as the kinematic substrate, showing that the category Top of (compactly generated) topological spaces embeds fully faithfully into condensed sets, so that "manifold" becomes a special, faithfully retained case of "condensed space." Part II [2] placed *observables* before geometry: it built a condensed sheaf of observable algebras on the pro-étale site, proved a descent (gluing) theorem for the algebra side, and — crucially for the present paper — proved that the *state* side fails descent, with the failure measured by a first cohomology class. The present Part III takes information itself as the primitive object and studies that failure as its own subject: cohomology as the measure of the obstruction to gluing informational data.

1.2 The hinge of the series

This paper is the *hinge* between the established first half of the program and its speculative second half, and it is written to be scrupulous about exactly that. Everything up to and including Part II can be read as rigorous mathematics dressed in physical vocabulary. Everything from Part IV onward asks the condensed machinery to do genuine physical work — to furnish fields without a background manifold, to reconstruct geometry as a realization functor, to model gravity as the gluing of informational structures. Part III sits precisely on the seam. Its *established* content is a faithful, citable account of cohomology as obstruction theory and of the quantum-information theorems (marginal problem, monogamy, holographic quantum error correction) that make "information glues badly" a precise statement. Its *speculative* content is a single, sharply stated proposal — that entanglement *is* a condensed

Čech cocycle, and that geometry is the moduli of informational compatibility — offered as a conjecture and never as a theorem.

1.3 Information as the primitive object

The methodological inversion of Part II was: observables before geometry. Part III pushes the inversion one step further: *information before observables*. Rather than beginning with a Hilbert space of states of some pre-given system and asking how information is stored in it, we take as primitive a system of local informational data — reduced density matrices, local expectation functionals, local subalgebras and their states — indexed over a site of "contexts," and ask under what conditions this local data glues to a global object. The answer, established rigorously in Part II and recalled here, is: *not always*, and the obstruction is entanglement. The novelty of Part III is to name the obstruction correctly. Obstructions to gluing local sections of a (pre)sheaf have a canonical mathematical measure — Čech and derived-functor cohomology — and, in the condensed setting where the coefficient objects are condensed abelian groups or condensed vector spaces, a canonical home — the derived ∞-category $\mathcal{D}(\mathrm{Cond}(\mathrm{Ab}))$. This paper is the systematic development of that identification, kept honest by a strict labelling discipline.

1.4 The epistemic discipline: EST, HEU, SPEC

Following the whole series, every substantive assertion carries one of three status labels.

- [EST] — *established*: rigorous, citable mathematics, or a rigorously proved physics-adjacent theorem (within its own stated hypotheses).
- [HEU] — *heuristic*: a physically motivated dictionary entry or interpretive reading, not itself a theorem.
- [SPEC] — *speculative*: an ontological or physical hypothesis of this program, with no established precedent.

The labels obey a *worst-component-wins* composition law: any chain of reasoning is capped at the status of its weakest link [2, 22]. Concretely, the mathematics of cohomology and condensed vector spaces is [EST]; the reading of $H^{\geq 1}$ as "protected information" is [HEU]; and the two headline claims — "entanglement *is* sheaf incompatibility" and "geometry emerges from informational compatibility" — are [SPEC], and remain [SPEC] no matter how much [EST] machinery is assembled beneath them. We restate this in Section 11 and tabulate it in Table 1.

1.5 Contributions and outline

The contributions of this paper are, by register:

Established (expository/structural). (i) A clean statement (Section 3) of Čech and derived-functor cohomology as the obstruction theory of gluing, specialized to the condensed pro-étale site, with the connecting map δ of Part II identified as the universal obstruction. (ii)

A self-contained account (Section 4) of why condensed (solid/liquid) vector spaces and their derived category are the correct home for state-valued cohomology, including the failure of naive topological vector spaces to form an abelian category. (iii) A worked, fully rigorous bridge between cohomology and quantum information (Section 5): the toric/surface code, whose logical operators are classified by $H_1(\Sigma; \mathbb{Z}/2)$ and H^1, and whose code distance is a systole — "topological data = protected information" as a theorem, not a slogan. (iv) A precise account (Section 6) of the quantum marginal problem, monogamy of entanglement, and strong subadditivity as the established obstructions to gluing states, and their reading as 0-th and 1-st Čech data. (v) A rigorous account (Section 8) of holographic quantum error correction — the HaPPY code, complementary recovery, entanglement-wedge reconstruction, and the Ryu–Takayanagi formula — as toy models of informational state reconstruction.

Speculative (the paper's original proposal). (vi) A precise *conjecture* (Section 7), boxed and labelled [SPEC], that entanglement across a cover *is* the nonvanishing of a condensed Čech class attached to a system of local states, together with a careful contrast against what is established and an explicit statement of what would be required to make it a theorem. (vii) A boxed [SPEC] hypothesis (Section 9) that geometry is the moduli of informational compatibility, presented as the condensed-cohomological upgrade of the ER = EPR heuristic and inheriting (at best) that heuristic's status.

Section 2 recalls the condensed framework. Section 10 states the consolidated results and describes the accompanying Haskell formalization. Section 11 discusses non-identifications, limitations, and the interface handed to Parts IV–VII, and Section 12 concludes.

2 Mathematical Framework

We recall, compactly, the condensed and homological machinery. Definitions carried over from Parts I–II are cited rather than reproved; the reader is referred to [1, 2, 3].

2.1 The condensed site and condensed abelian groups

Definition 2.1 (Pro-étale site of a point; condensed sets)**.** [EST] Let ProFin be the category of profinite sets (compact, Hausdorff, totally disconnected spaces; equivalently cofiltered limits of finite discrete sets). The *pro-étale site of a point* $*_{\text{pro-ét}}$ is ProFin equipped with the topology whose covers are finite jointly surjective families $\{S_i \to S\}$. A *condensed set* is a sheaf $T : \mathrm{ProFin}^{\mathrm{op}} \to \mathrm{Set}$ (with the usual κ-smallness bookkeeping): $T(\emptyset) = *$, $T(S \sqcup S') = T(S) \times T(S')$, and for every surjection $S' \twoheadrightarrow S$ the sequence $T(S) \to T(S') \rightrightarrows T(S' \times_S S')$ is an equalizer. *Condensed abelian groups* Cond(Ab) are abelian-group objects therein; equivalently, sheaves of abelian groups on $*_{\text{pro-ét}}$ [3].

Theorem 2.2 (Basis by extremally disconnected sets)**.** [EST] *The extremally disconnected (Stonean) profinite sets — those in which the closure of every open set is open, equivalently the projective objects of* CompHaus *— form a basis for* $*_{\text{pro-ét}}$*. Every condensed set is determined by its values on Stonean objects, and on those objects the sheaf condition trivializes (finite products and split surjections). Consequently* Cond(Ab) *is a Grothendieck abelian category: it has all limits and colimits, a generator, and enough projectives (the free condensed abelian groups on Stonean sets) and injectives [3].*

The last sentence is the crucial technical fact for this paper: unlike topological abelian groups, condensed abelian groups form an abelian category, so the entire apparatus of homological algebra — kernels, cokernels, Ext, Tor, derived functors, derived categories — is available without pathology. This is precisely the repair we will need, one register up, for state-valued coefficients.

2.2 Solid modules and condensed vector spaces

Definition 2.3 (Solid abelian groups; condensed vector spaces)**.** [EST] A condensed abelian group M is *solid* if for every profinite set $S = \varprojlim_i S_i$ the natural map from the free condensed abelian group $\mathbb{Z}[S]$ to the "measures" completion $\mathbb{Z}[S]^{\blacksquare} := \varprojlim_i \mathbb{Z}[S_i]$ induces a bijection $\mathrm{Hom}(\mathbb{Z}[S]^{\blacksquare}, M) \cong \mathrm{Hom}(\mathbb{Z}[S], M)$ (informally: profinitely-indexed null sequences are uniquely summable in M). Solid abelian groups $\mathbb{Z}_{\blacksquare}$-Mod form a full abelian subcategory of Cond(Ab) closed under limits, colimits, and extensions, with a well-behaved derived category $\mathcal{D}(\mathbb{Z}_{\mathrm{solid}})$ [3, 5]. Over $\mathbb{R}$ or $\mathbb{C}$ the analogous role is played by the *p-liquid* vector spaces; we write Cond(Vect) generically for condensed $\mathbb{C}$-vector spaces and use "solid/liquid" when the completion condition matters.

Remark 2.4 (Why not topological vector spaces). [EST] Continuous linear maps of topological vector spaces do not form an abelian category: the map $\mathbb{R}_{\mathrm{disc}} \to \mathbb{R}$ is a continuous monomorphism and epimorphism that is not an isomorphism, so "kernel" and "cokernel" misbehave and Ext/Tor are ill-defined. Condensed, and specifically solid/liquid, modules repair exactly this: they are the smallest well-behaved abelian enlargement in which "complete topological vector space of states/distributions" lives without destroying homological algebra [3, 4]. Because the cohomology classes that will measure entanglement are *valued* in spaces of density-matrix data, and we intend to apply Ext, RHom, and derived tensor products to them, this repair is not optional bookkeeping but the reason the theory can be written at all.

2.3 Derived categories and stable ∞-categories

Definition 2.5 (The derived category of condensed coefficients)**.** [EST] Let $\mathfrak{A}$ be one of the abelian categories Cond(Ab), $\mathbb{Z}_{\blacksquare}$-Mod, or Cond(Vect). Its *derived* ∞*-category* $\mathcal{D}(\mathfrak{A})$ is the stable ∞-category obtained by inverting quasi-isomorphisms of cochain complexes in $\mathfrak{A}$; its homotopy category is triangulated (Verdier). It carries a t-structure whose heart is $\mathfrak{A}$, exact triangles $X \to Y \to Z \to X[1]$, homotopy limits and colimits, and derived functors $R\Gamma$, $R\mathrm{Hom}$, $\otimes^L$, with $H^i R\mathrm{Hom}(X, Y) = \mathrm{Ext}^i(X, Y)$ [7, 6].

This is the ambient category of "informational obstructions" for the rest of the paper: a system of local states with incompatible overlaps is an object of $\mathcal{D}(\mathrm{Cond}(\mathrm{Vect}))$ whose higher cohomology is the measured incompatibility.

2.4 Notational conventions

We write Ctx for a small site of "contexts" (measurement contexts, regions, or condensed test objects), always understood to be either a full subcategory of $*_{\text{pro-ét}}$ or the poset of regions of Part II; U, V for its objects; $\{U_i \to U\}$ for covers; $\mathcal{O}, \mathcal{F}, \mathcal{G}$ for sheaves; ω for a global state and ρ for a density matrix; and $\mathcal{D}(-)$ for derived categories. The symbol $\check{\mathrm{H}}^p$ denotes Čech cohomology; H^p denotes sheaf (derived-functor) cohomology. All Hilbert spaces are finite-dimensional except where a condensed/solid completion is explicitly invoked.

3 Cohomology as the Measure of Gluing Obstruction

This section is entirely [EST]. It develops the one mathematical idea the whole paper turns on: cohomology is the exact obstruction to passing from local data to global data. We give it three times — Čech, derived-functor, and via hypercovers on the condensed site — and prove they agree where they should.

3.1 The Čech complex of a cover

Definition 3.1 (Čech complex). [EST] Let $\mathcal{F}$ be a sheaf of abelian groups on a site and $\mathfrak{U} = \{U_i \to U\}_{i \in I}$ a cover. Write $U_{i_0 \cdots i_p} := U_{i_0} \times_U \cdots \times_U U_{i_p}$. The *Čech complex* $\check{\mathrm{H}}^\bullet(\mathfrak{U}, \mathcal{F})$ is

$$\prod_{i_0} \mathcal{F}(U_{i_0}) \xrightarrow{d^0} \prod_{i_0 < i_1} \mathcal{F}(U_{i_0 i_1}) \xrightarrow{d^1} \prod_{i_0 < i_1 < i_2} \mathcal{F}(U_{i_0 i_1 i_2}) \xrightarrow{d^2} \cdots, \tag{1}$$

with differential $(d^p s)_{i_0 \cdots i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \, s_{i_0 \cdots \widehat{i_k} \cdots i_{p+1}}\big|_{U_{i_0 \cdots i_{p+1}}}$. Its cohomology is the *Čech cohomology* $\check{\mathrm{H}}^p(\mathfrak{U}, \mathcal{F})$.

Proposition 3.2 (Čech degrees 0 and 1 are exactly gluing data). [EST] *For any sheaf $\mathcal{F}$ and cover $\mathfrak{U}$:*

(i) $\check{\mathrm{H}}^0(\mathfrak{U}, \mathcal{F}) = \ker d^0 = \mathcal{F}(U)$*: a* 0*-cocycle is a family of local sections that agree on overlaps, i.e. precisely a global section (descent in degree* 0*).*

(ii) $\check{\mathrm{H}}^1(\mathfrak{U}, \mathcal{F})$ *classifies* $\mathcal{F}$*-torsors trivialized on* $\mathfrak{U}$*: a* 1*-cocycle* (g_{ij}) *satisfies* $g_{ij} + g_{jk} = g_{ik}$ *on triple overlaps, a coboundary is* $g_{ij} = h_i - h_j$*, and* $\check{\mathrm{H}}^1$ *is cocycles modulo coboundaries. A class vanishes iff the local data glues; the class* is *the obstruction to gluing.*

Proof. (i) $d^0(s)_{ij} = s_j|_{U_{ij}} - s_i|_{U_{ij}}$, so $\ker d^0$ is the set of families agreeing on overlaps; the sheaf axiom identifies this equalizer with $\mathcal{F}(U)$. (ii) $d^1(g)_{ijk} = g_{jk} - g_{ik} + g_{ij}$ vanishes iff (g_{ij}) is a cocycle; $d^0(h)_{ij} = h_j - h_i$ are the coboundaries; the quotient is $\check{\mathrm{H}}^1$. The identification with torsors and the vanishing criterion is the torsor-classification proposition of Part II (Giraud's nonabelian H^1 in the abelian case) [2, 10, 8]. □

Proposition 3.2 is the entire conceptual content in miniature. Degree 0 is "do the local observations glue?"; the answer lives in the sheaf axiom. Degree 1 is "*by how much* do they fail to glue?"; the answer is a cohomology class. Everything below is the systematic, condensed, and quantum-informational elaboration of these two lines.

3.2 Derived-functor cohomology and the universal obstruction

Definition 3.3 (Sheaf cohomology). [EST] For $\mathcal{F} \in \mathrm{Cond(Ab)}$ (or a sheaf of abelian groups on any site), the *sheaf cohomology* of U is $H^p(U, \mathcal{F}) := H^p R\Gamma(U, \mathcal{F})$, the cohomology of the right-derived global-sections functor, computed by any injective (or, on $*_{\text{pro-ét}}$, any Stonean-projective) resolution.

Proposition 3.4 (The connecting map is the universal gluing obstruction). [EST] *Let* $0 \to \mathcal{G} \to \mathcal{F} \to \mathcal{Q} \to 0$ *be a short exact sequence in* $\mathrm{Cond(Ab)}$. *The induced long exact sequence*

$$0 \to \mathcal{G}(U) \to \mathcal{F}(U) \to \mathcal{Q}(U) \xrightarrow{\delta} H^1(U, \mathcal{G}) \to H^1(U, \mathcal{F}) \to \cdots \tag{2}$$

identifies, for a global section $\bar{s} \in \mathcal{Q}(U)$, the obstruction to lifting $\bar{s}$ to $\mathcal{F}(U)$ with the single class $\delta(\bar{s}) \in H^1(U, \mathcal{G})$: the lift exists iff $\delta(\bar{s}) = 0$. Moreover $\delta(\bar{s})$ is represented by the Čech 1*-cocycle of differences of any system of local lifts on any cover.*

Proof. Exactness of (2) at $\mathcal{Q}(U)$ gives $\bar{s} \in \mathrm{im}(\mathcal{F}(U) \to \mathcal{Q}(U))$ iff $\bar{s} \in \ker \delta$, i.e. iff $\delta(\bar{s}) = 0$. The Čech representative: choose lifts $s_i \in \mathcal{F}(U_i)$ of $\bar{s}|_{U_i}$ on a cover; on overlaps $s_i - s_j$ maps to 0 in $\mathcal{Q}$, hence lies in $\mathcal{G}(U_{ij})$ and forms a 1-cocycle whose class is $\delta(\bar{s})$, independent of the choices by Proposition 3.2(ii). This is the standard construction of the connecting homomorphism [8, 9, 7]. □

This is the sharp form of the sentence carried over from Part II: whenever an observable is defined only *modulo* a subsheaf — a quotient/ratio observable, a state defined up to local gauge, a system defined only up to its marginals — the failure of local data to determine global data is a single, canonical cohomology class. It is "the" obstruction because it is universal: it depends only on the short exact sequence, not on the choices of local lifts.

3.3 Computation on the condensed site: Stonean hypercovers

Proposition 3.5 (Derived cohomology via Stonean hypercovers). [EST] *Let* $\mathcal{O} \in \mathrm{Cond(Ab)}$ *and let $S_\bullet \to U$ be a hypercover all of whose levels S_n are extremally disconnected. Then*

$$R\Gamma(U, \mathcal{O}) \;\simeq\; \mathrm{Tot}\left[\, \mathcal{O}(S_0) \to \mathcal{O}(S_1) \to \mathcal{O}(S_2) \to \cdots \,\right], \qquad H^p(U, \mathcal{O}) \cong H^p\big(\mathcal{O}(S_\bullet)\big), \tag{3}$$

the cohomology of the normalized cochain complex of the cosimplicial abelian group $\mathcal{O}(S_\bullet)$.

Proof. Each S_n is Stonean, hence $\mathcal{O}$-acyclic ($H^q(S_n, \mathcal{O}) = 0$ for $q > 0$), because Stonean objects are projective in $*_{\text{pro-ét}}$ [3]. The Bousfield–Kan descent spectral sequence of $\mathcal{O}(S_\bullet)$ has $E_1^{p,q} = H^q(S_p, \mathcal{O})$, concentrated in the row $q = 0$, and collapses onto the normalized complex of $\mathcal{O}(S_\bullet)$, converging to $H^p(U, \mathcal{O})$; equivalently $S_\bullet$ is a projective (acyclic) resolution of U, so evaluating $\mathcal{O}$ computes $R\Gamma$ [8, 7]. This is the derived-descent proposition of Part II [2]. □

Remark 3.6 (Hypercover, not Čech nerve). [EST] The resolution must be a hypercover with *every* level Stonean, not the Čech nerve of a single Stonean cover: fibre products of infinite Stonean spaces are generically not Stonean, so the higher Čech terms need not be acyclic and Čech cohomology can differ from derived cohomology [2, 3]. This is the condensed avatar of the classical fact that Čech and derived-functor cohomology agree only under acyclicity hypotheses (e.g. good covers). Where we write Ȟ below we mean Čech cohomology of an explicit finite cover — always well defined and the directly "operational" object — and we flag where it may differ from $H^\bullet$.

4 Condensed Vector Spaces and Derived Categories as the Home for Obstructions

We now explain, still in the [EST] register, why the coefficients of the cohomology that will measure informational obstructions belong in condensed (solid/liquid) vector spaces and their derived category, rather than in Hilbert spaces or topological vector spaces.

4.1 States and density matrices as condensed-linear data

Definition 4.1 (State datum over a context site). [EST] Let Ctx be a context site and let $\mathfrak{A} : \mathsf{Ctx}^{\mathrm{op}} \to C^*\mathrm{Alg}$ be a (co)sheaf of observable algebras as in Part II [2]. A *state datum* is a presheaf $\mathcal{O}_\omega : \mathsf{Ctx}^{\mathrm{op}} \to \mathrm{Cond}(\mathrm{Vect})$ sending U to the space of normal states (density operators) on $\mathfrak{A}(U)$, with restriction along $V \subseteq U$ given by the partial trace / marginalization $\rho \mapsto \rho|_{\mathfrak{A}(V)}$. For finite-dimensional local algebras $\mathfrak{A}(U) = B(\mathcal{H}_U)$ the states form the convex set of density matrices ρ with $\rho \geq 0$, $\mathrm{tr}\,\rho = 1$: an *affine* hyperplane inside the (solid) $\mathbb{C}$-vector space $\mathcal{V}(U)$ of trace-class Hermitian operators on $\mathcal{H}_U$. When we form Čech complexes and pass to the derived category below, the coefficients are the ambient *linear* spaces $\mathcal{V}(U)$, not the affine state sets: differences of two states are trace-*zero* Hermitian operators, and it is in $\mathcal{V}(-)$ (an abelian, in fact solid, coefficient category) that all homological algebra takes place, with the physical states recorded as the $\mathrm{tr} = 1$ affine subset.

The point of Definition 4.1 is that a state is *linear* data (a functional, or a trace-class operator), and linear data over a profinitely-indexed base wants to live in a category where infinite convex/linear combinations indexed over a profinite set of "branches" or "contexts" are controlled. That is exactly what solid and liquid modules provide (Definition 2.3); the naive alternative — trace-class operators with the trace-norm topology — is a topological vector space and hence, by Remark 2.4, homologically pathological.

Proposition 4.2 (The marginalization map is condensed-linear but not multiplicative). [EST] *Partial trace* $\mathrm{tr}_{V^c} : B(\mathcal{H}_U) \to B(\mathcal{H}_V)$ *is a completely positive, trace-preserving,* $\mathbb{C}$-*linear map, continuous for the trace norm, hence a morphism of condensed vector spaces on state data. It is* not *an algebra homomorphism:* $\mathrm{tr}_{V^c}(ab) \neq \mathrm{tr}_{V^c}(a)\,\mathrm{tr}_{V^c}(b)$ *in general. Consequently the assignment* $U \mapsto \mathcal{O}_\omega(U)$ *is a presheaf of* sets/affine spaces *that is linear on each fibre but does* not *inherit descent from the algebra (co)sheaf.*

Proof. Complete positivity and trace preservation of partial trace are standard; linearity and trace-norm continuity make it a $\mathrm{Cond}(\mathbb{C})$-morphism. Non-multiplicativity is exhibited by any entangled ρ: for a Bell state $\rho = |\Phi^+\rangle\langle\Phi^+|$ on $\mathbb{C}^2 \otimes \mathbb{C}^2$, both marginals are $\frac{1}{2}I$, yet $\rho \neq \frac{1}{2}I \otimes \frac{1}{2}I$, so the marginals do not multiply back to the joint state. Functoriality of restriction (Proposition 3.2) then makes $\mathcal{O}_\omega$ a presheaf, but Proposition 6.1 below shows it is not a sheaf. □

4.2 The state complex and its cohomology

Given a cover $\mathfrak{U} = \{U_i \to U\}$ and the state presheaf $\mathcal{O}_\omega$, the Čech complex (1) is formed not in the affine state sets but in the ambient linear coefficient spaces $\mathcal{V}(U_{i_0\cdots i_p})$ of trace-class Hermitian operators (Definition 4.1), each a solid/liquid condensed vector space; the differentials land in the trace-zero subspaces. We package the failure of descent as an object of the derived category.

Definition 4.3 (Derived state object)**.** [EST] The *derived state object* of U on the cover $\mathfrak{U}$ is $R\Gamma(\mathfrak{U}, \mathcal{O}_\omega) \in \mathcal{D}(\mathrm{Cond}(\mathrm{Vect}))$, the totalization of the cosimplicial condensed vector space $\mathcal{O}_\omega(U_\bullet)$, with cohomology $\check{\mathrm{H}}^p(\mathfrak{U}, \mathcal{O}_\omega)$. Degree 0 recovers the globally consistent states (those agreeing with all marginals); degree 1 measures the first obstruction to assembling overlap-compatible marginals into a global state.

Remark 4.4 (Why the derived category is unavoidable). [HEU] Three forces, exactly as in Part II, force the passage from $\mathcal{O}_\omega(U)$ to $R\Gamma(\mathfrak{U}, \mathcal{O}_\omega)$: (i) states are quotient/affine data (defined up to local gauge and normalization), so sheafification manufactures an H^1; (ii) locally-standard-but-globally-twisted state data are $\check{\mathrm{H}}^1$-torsors; (iii) the coefficient spaces are condensed anima with internal higher structure, so "agree on overlaps" means "agree up to coherent homotopy," which is descent in a stable ∞-category and is computed by $R\Gamma$. The [HEU] content is the interpretive claim that $H^{\geq 1}$ of the state complex is "information stored nonlocally across the cover"; the underlying homological algebra is [EST].

5 A Rigorous Bridge: The Surface Code and $H_1(\Sigma; \mathbb{Z}/2)$

Before proposing anything speculative, we present the one place where "cohomology = protected quantum information" is a *theorem*. The toric/surface code makes the identification exact and is the load-bearing worked example the paper's speculation is calibrated against.

5.1 The toric code and its logical operators

Definition 5.1 (Toric/surface code)**.** [EST] Let Σ be a closed orientable surface with a cellulation (a $\mathbb{Z}/2$ chain complex $C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$ with C_1 spanned by edges, on each of which sits a qubit). The *toric code* is the stabilizer code with vertex operators $A_v = \prod_{e \ni v} X_e$ (one per vertex v) and plaquette operators $B_p = \prod_{e \in \partial p} Z_e$ (one per face p). The code space is the simultaneous $+1$ eigenspace of all A_v, B_p [21, 16].

Theorem 5.2 (Logical operators are homology/cohomology classes)**.** [EST] *For the toric code on a closed orientable surface Σ of genus g:*

(i) The Z-type logical operators, modulo stabilizers, are in canonical bijection with $H_1(\Sigma; \mathbb{Z}/2)$; the X-type logical operators with $H^1(\Sigma; \mathbb{Z}/2)$. Both have dimension $2g$, so the code encodes $k = 2g$ logical qubits: $\dim($*code space*$) = 2^{2g}$.

(ii) The logical Z and X operators pair by the evaluation map $H_1 \times H^1 \to \mathbb{Z}/2$ (equivalently, the intersection form on H_1 via Poincaré duality), reproducing the canonical symplectic (anticommutation) structure of the $2g$ logical qubits.

(iii) The code distance equals the systole*: the length of the shortest homologically nontrivial cycle. Protected information is invisible to any local operator supported on a contractible (homologically trivial) region.*

Proof sketch. A Z-string $\prod_{e\in\gamma} Z_e$ commutes with every vertex operator A_v iff γ is a cycle ($\partial_1\gamma = 0$), and it is a product of plaquette stabilizers iff γ is a boundary ($\gamma \in \operatorname{im}\partial_2$); hence nontrivial Z-logicals are $\ker\partial_1/\operatorname{im}\partial_2 = H_1(\Sigma;\mathbb{Z}/2)$. Dually, X-strings live on the dual cellulation and give $H^1(\Sigma;\mathbb{Z}/2)$. Poincaré duality identifies the two and the cup/intersection product supplies the symplectic pairing; the minimal weight of a nontrivial logical is the combinatorial systole. See [21, 16] and references therein. □

Remark 5.3 (This is the calibration point). [EST]/[HEU]. Theorem 5.2 is completely rigorous and is the honest content of the slogan "topological/cohomological data = protected quantum information." It is *finite*, *ordinary* (not condensed) cohomology of a *genuine* space Σ. Two features generalize cleanly and one does not. Generalizing cleanly: (a) the *form* "information protected against local erasure = a cohomology class"; (b) the pairing of complementary logicals by an intersection form. *Not* generalizing for free: the existence of an actual surface Σ carrying the cohomology. The speculative proposal of Section 7 is precisely to *replace* Σ by a condensed site of informational contexts and to ask whether entanglement is the cohomology of *that*. We keep the two apart scrupulously: Theorem 5.2 is a theorem about a space; the conjecture is about whether a space is needed at all.

6 The Quantum Marginal Problem as a Descent Problem

We now assemble the established quantum-information facts that make "states glue badly" precise, and read them cohomologically *as far as, but no further than,* is rigorous.

6.1 The state presheaf fails descent

Proposition 6.1 (Recalled from Part II: the entanglement obstruction)**.** [EST] *The state presheaf $\mathcal{O}_\omega$ (Definition 4.1) is in general neither separated nor a sheaf:*

(i) Not separated*: for spacelike/tensor-separated R_1, R_2, an entangled global state ω and the product $\omega|_{R_1} \otimes \omega|_{R_2}$ of its marginals agree on the cover $\{R_1, R_2\}$ yet differ globally — distinct global states with identical local data.*

(ii) Not a sheaf*: a family of pairwise-compatible local states need not admit any globally positive extension; the set of admissible global states with prescribed marginals is a (possibly empty, generically positive-dimensional) convex set.*

The obstruction to descent of $\mathcal{O}_\omega$ across a cover is the signature of entanglement across that cover [2, 11].

Proof. This is the state-presheaf (entanglement-obstruction) proposition of Part II [2]; the Bell state witnesses (i) and the quantum marginal (representability) problem [11] witnesses (ii). □

6.2 What is established: marginals, monogamy, subadditivity

We list, all [EST], the rigorous constraints that any candidate "cohomology of an entangled state" must be consistent with. These are the guardrails against overclaiming.

Theorem 6.2 (Quantum marginal / representability problem). [EST] *Given local density matrices ρ_S on subsystems S from a cover, deciding whether a global ρ with $\mathrm{tr}_{S^c}\,\rho = \rho_S$ exists is the* quantum marginal problem*; for overlapping marginals it is QMA-complete in general (Liu; Klyachko for the non-overlapping spectral case gives an explicit polytope of admissible spectra) [11]. The compatible global states, when they exist, form a convex set.*

Theorem 6.3 (Monogamy of entanglement). [EST] *Entanglement cannot be freely shared. For three qubits A, B, C the Coffman–Kundu–Wootters inequality holds, $\tau_{A|BC} \geq \tau_{AB} + \tau_{AC}$, where τ is the tangle (squared concurrence) [11, 18]. If A is maximally entangled with B it is necessarily product with C.*

Theorem 6.4 (Strong subadditivity of von Neumann entropy). [EST] *For any tripartite state ρ_{ABC}, $S(AB) + S(BC) \geq S(ABC) + S(B)$, where $S(\rho) = -\mathrm{tr}\,\rho \log \rho$ (Lieb–Ruskai). Equality holds iff ρ_{ABC} is a quantum Markov chain $A - B - C$, i.e. recoverable from its AB and BC marginals by a Petz map [16].*

Remark 6.5 (Reading these as Čech data, honestly). [HEU] These theorems supply the exact operational meaning of Čech degrees 0 and 1 for state data, and no more. A 0-*cochain* is a choice of local marginal on each cover element; the 0-*cocycle* condition is overlap-compatibility of marginals — and Theorem 6.2 says determining whether a 0-cocycle exists (a global state with those marginals) is already hard, while Proposition 6.1(ii) says the fibre of "global state" over a compatible 0-cocycle is a convex set, not a point. Theorem 6.4's equality case — quantum Markov chains — is precisely the "locally trivial" case where the B-overlap already glues A and C, the informational analogue of a sheaf that happens to satisfy descent. Theorem 6.3 is the quantitative statement that the gluing obstruction cannot be shared: a large "H^0-agreement" of A with B forces triviality of A–C correlations. We will *use* these as constraints on any proposed cohomology, and we will not assert more than they license.

7 Entanglement as Sheaf and Cosheaf Compatibility

This section states the paper's central original proposal. Per §0.2 and §4.4 of the program's knowledge base and our own literature search, *no precedent exists* in the literature combining condensed cohomology with entanglement; the following is therefore offered as a [SPEC] conjecture, not a synthesis of known results, and is framed to be falsifiable in principle. We first fix what would have to be true, then box the conjecture, then say plainly what is *not* claimed.

7.1 Two obstructions: marginal consistency (sheaf) and reconstruction (cosheaf)

Fix a context site Ctx, a cover $\mathfrak{U} = \{U_i \to U\}$, and the state presheaf $\mathcal{O}_\omega$ valued (linearly) in the condensed vector spaces $\mathcal{V}(-)$ of Definition 4.1. There are *two* distinct obstructions, living on *overlaps* and on *unions* respectively, and it is essential — and, per the round-one referee, mathematically load-bearing — not to conflate them. They correspond precisely to the sheaf-versus-cosheaf fork of Part II [2]: kinematic marginal data *restricts* (a sheaf, limit-gluing on intersections), while joint states *compose* (a cosheaf, colimit-gluing on unions).

Obstruction I (overlaps, sheaf). A *system of local states* is a 0-cochain $\rho_\bullet = (\rho_i)$, $\rho_i \in \mathcal{O}_\omega(U_i)$. Its *overlap discrepancy* is the 1-cochain of the Čech complex (1), with the sign convention of d^0 fixed in Definition 3.1,

$$(d^0\rho)_{ij} \;:=\; \rho_j\big|_{U_{ij}} - \rho_i\big|_{U_{ij}} \;\in\; \mathcal{V}(U_{ij}), \tag{4}$$

the difference of marginals on the overlap. The system is *marginally consistent* if $d^0\rho = 0$. This is the sheaf side and it is the entry point of the quantum marginal problem: by Proposition 6.1 and Theorem 6.2, marginal consistency does not imply the existence or uniqueness of a global state.

Obstruction II (unions, cosheaf). The genuinely new object measures whether a global state is *reconstructible* from its parts, and — crucially — it does *not* live on overlaps. As the referee observed, a difference of two states *on the overlap* U_{ij} would be identically zero for any recombination that preserves marginals (a Petz map does), so it could never see entanglement. The obstruction is instead a difference of two states *on the union* $U_i \cup U_j$, and is therefore a datum of a *cosheaf*/factorization complex, not of the sheaf Čech complex.

Definition 7.1 (Entanglement chain of a global state)**.** [SPEC] Let ω be a global state and $\mathfrak{U}$ a cover. For each pair i, j let $\star_{ij}$ denote the *free/Markov recombination* on the union: the maximal-entropy (Petz) reconstruction of a state on $U_i \cup U_j$ from the two one-sided marginals $\omega|_{U_i}$, $\omega|_{U_j}$ glued along their shared subsystem U_{ij} (the product state when $U_{ij} = \emptyset$). Define the *entanglement* 1*-chain* by

$$c_\omega(\mathfrak{U})_{ij} \;:=\; \omega|_{U_i \cup U_j} \;-\; \big(\omega|_{U_i}\big) \star_{ij} \big(\omega|_{U_j}\big) \;\in\; \mathcal{V}(U_i \cup U_j), \tag{5}$$

the difference, *on the union*, between the true joint state and its free/Markov recombination. Being a difference of two states with equal marginals, $c_\omega(\mathfrak{U})_{ij}$ is a trace-zero Hermitian element of the condensed vector space $\mathcal{V}(U_i \cup U_j)$; it vanishes iff ω restricted to $U_i \cup U_j$ is the Markov reconstruction of its parts.

Example 7.2 (The Bell chain is nonzero)**.** [EST] For ω the Bell state on $U_1 = A$, $U_2 = B$ with $U_{12} = \emptyset$, both one-sided marginals are $\frac{1}{2}I$ and $\star_{12}$ is the product $\frac{1}{2}I \otimes \frac{1}{2}I = \frac{1}{4}I$; the true joint state is the rank-one Bell projector, so $c_\omega = |\Phi^+\rangle\langle\Phi^+| - \frac{1}{4}I \neq 0$. For a product state the same difference vanishes. Thus (5) is nonzero *exactly* for the entangled state — the behaviour (4) could not exhibit. (This is the finite computation performed in `Marginal.hs`.)

The recombination $\star_{ij}$ is defined on the locus where a Petz/Markov reconstruction on the union exists; off that locus c_ω must be read in the derived sense, as a class in the cosheaf homology $R\Gamma_c(\mathfrak{U}, \mathcal{V}) \in \mathcal{D}(\mathrm{Cond}(\mathrm{Vect}))$. This is the technical reason the proposal is *condensed*-homological and lives naturally in $\mathcal{D}(\mathrm{Cond}(\mathrm{Vect}))$: the union-indexed differences do not fit a plain vector space once $\star_{ij}$ degenerates.

7.2 The central conjecture

[SPEC] Conjecture: Entanglement is a condensed cosheaf class

Conjecture 7.3 (Entanglement–reconstruction correspondence)**.** [SPEC] There is a condensed cosheaf (factorization) homology theory $H^{\mathrm{Cond}}_{\bullet}(-, \mathcal{V})$ valued in $\mathcal{D}(\mathrm{Cond}(\mathrm{Vect}))$, functorial in the context site and the global state, in which (5) is a 1-chain, such that for every cover $\mathfrak{U}$ of a context U:

(A) the class $[\, c_\omega(\mathfrak{U}) \,] \in H^{\mathrm{Cond}}_1(\mathfrak{U}, \mathcal{V})$ vanishes *if and only if* the global state ω is reconstructible from its one-sided marginals along $\mathfrak{U}$ (equivalently, the restriction of ω to the cover is a quantum Markov network for $\mathfrak{U}$); and

(B) a suitable norm/rank invariant of $[\, c_\omega(\mathfrak{U}) \,]$ is monotone under local operations and classical communication (LOCC) restricted to the cover, and obeys the monogamy and strong-subadditivity constraints of Theorems 6.3 and 6.4 as inequalities among the classes on refinements of $\mathfrak{U}$.

In slogan form: *entanglement across a cover is the nonvanishing of a condensed cosheaf class of the state data on unions, and separable (Markov-reconstructible) states are exactly the chain-trivial ones.* The dual sheaf obstruction (4) on overlaps governs, instead, the marginal problem; the two are related by the sheaf/cosheaf duality of Part II.

7.3 What is claimed, and what is not

We are deliberately explicit, because this is the seam of the whole series.

What is established (and used above, not conjectured). The state presheaf fails descent (Proposition 6.1, [EST]); the failure is measured, in the abelian/derived setting, by a connecting map into H^1 (Proposition 3.4, [EST]); the quantum-Markov (equality-SSA) states are exactly the marginally reconstructible ones (Theorem 6.4, [EST]); and monogamy constrains how correlation can be distributed (Theorem 6.3, [EST]). These are the facts Conjecture 7.3 is designed to organize.

What is conjectured (and must not be read as proved). That the informal "entanglement chain" (5) is (i) the 1-chain of an honest condensed cosheaf (factorization) homology theory on unions, with the functoriality and target category asserted; (ii) that its vanishing is *exactly* Markov reconstructibility (the "if" direction is essentially Theorem 6.4; the "only if" at the level of a single canonical class is the substantive open claim); and (iii) that a class-level invariant is an LOCC monotone. None of (i)–(iii) is a theorem here. In particular the existence of the theory $H^{\mathrm{Cond}}_{\bullet}$ with a well-defined global $\star_{ij}$ off the Petz-recoverable locus is open.

What is explicitly not *claimed.* We do *not* claim that entanglement entropy equals any dimension of a cohomology group; entropy is a real-valued, basis-independent, generically irrational invariant, whereas cohomological dimensions are integers, so at most a *filtration* or *rank* of c_ω (not the entropy itself) can be cohomological. We do *not* claim a general algorithm computing c_ω (the marginal problem is QMA-complete, Theorem 6.2, so any such theory inherits that hardness). And we do *not* claim priority for "entanglement is cohomological" as a vague idea — contextuality has a well-known Čech-cohomological obstruction theory (Abramsky–Brandenburger and successors) — but rather for the specific *condensed* and *state-reconstruction* formulation above, whose novelty is the use of solid/liquid coefficients so that Ext/RHom of the state complex is well defined.

Remark 7.4 (Relation to contextuality cohomology). [EST]/[SPEC]. The sheaf-theoretic account of *contextuality* (Abramsky–Brandenburger [20]) already expresses the impossibility of a global joint distribution over locally consistent measurement statistics as a Čech obstruction, and is a genuine [EST] precedent for "no global section = nonlocality." Our proposal differs in two ways that we do not conflate: (a) its coefficients are *condensed vector spaces of quantum states*, not $\mathbb{Z}/2$ or the semiring of the empirical model, so it targets entanglement of the state, not contextuality of measurement outcomes; and (b) it lives on the *condensed* pro-étale site, so the coefficient category supports the derived functors the reconstruction story (Section 8) needs. The relation to contextuality cohomology is a [SPEC] conjecture (that one specializes to the other on classical covers), not an identification.

7.4 The ER = EPR lineage

[HEU] Lineage, not evidence

Conjecture 7.3 is the condensed-cohomological upgrade of the ER = EPR heuristic [18, 19]: "entanglement is (a geometric) connection" becomes "entanglement is a cocycle." We flag emphatically that ER = EPR is itself only [HEU]/[SPEC] even in its native AdS/CFT setting, and that citing it is stating a *lineage*, not adducing evidence. Under the worst-component-wins discipline, any claim resting on this lineage is capped at [SPEC].

8 Information Flow and State Reconstruction

Having isolated the obstruction, we turn to the constructive side: when *can* global information be recovered from local data? The rigorous answer, within toy models, is the theory of quantum error correction and holographic reconstruction. Everything in this section is [EST] *as a statement about the toy models*; its extrapolation to a general mechanism of emergent spacetime is [HEU]/[SPEC] and flagged as such.

8.1 Erasure correction as a gluing statement

Definition 8.1 (Quantum error-correcting code; erasure)**.** [EST] A code is an isometry $V : \mathcal{H}_{\text{logical}} \hookrightarrow \mathcal{H}_{\text{physical}} = \bigotimes_i \mathcal{H}_i$. A region $A \subseteq \{i\}$ is *correctable* against erasure if there is a recovery channel $\mathbb{R}_A$ with $\mathbb{R}_A \circ \mathcal{E}_{A^c} \circ \mathcal{V}_{\text{enc}} = \mathcal{V}_{\text{enc}}$, where $\mathcal{V}_{\text{enc}}(\rho) = V\rho V^\dagger$ is the CPTP encoding channel of the isometry V and $\mathcal{E}_{A^c}$ traces out ("erases") the complement A^c. Equivalently (Knill–Laflamme), writing $P = VV^\dagger$ for the code projector on the physical space, erasure of A^c is correctable iff $P\,O_{A^c}\,P \propto P$ for all operators O_{A^c} supported on A^c (equivalently $V^\dagger O_{A^c} V \propto I_{\text{logical}}$).

Proposition 8.2 (Complementary recovery)**.** [EST] *For a code with the complementary-recovery property, every bulk/logical operator is represented* either *in the subalgebra $\mathfrak{A}_A$ of a boundary region A* or *in that of its complement $\mathfrak{A}_{A^c}$, and the two representations agree on the code space. Thus a logical operator is an equivalence class of boundary representatives glued across the $\{A, A^c\}$ cover — a section of a sheaf of "reconstructions" whose overlaps are controlled by the code [15, 16].*

Proof sketch. Complementary recovery is the statement that the code subspace supports an operator-algebra decomposition in which $\mathfrak{A}'_A = \mathfrak{A}_{A^c}$ on the code space; a logical operator, being in the center, is representable on either side, and the two representatives act identically on code states by definition of the recovery channels [15, 16]. This is the operator-algebra quantum error correction of Harlow's reconstruction theorem. □

Proposition 8.2 is the precise, [EST] sense in which "measurement reconstructs an object without that object having independent existence apart from its representations" — the paradigm the whole series orbits. The bulk operator is not a thing plus its shadows; it *is* the compatible system of boundary shadows, i.e. a 0-cocycle for the reconstruction sheaf.

8.2 The HaPPY code and holographic cohomology of information

Theorem 8.3 (HaPPY code properties)**.** [EST] *The HaPPY holographic code — a tensor network of perfect tensors on a hyperbolic tessellation — is an isometry $V : \mathcal{H}_{\text{bulk}} \hookrightarrow \mathcal{H}_{\text{boundary}}$ such that: (i) bulk operators in the entanglement wedge of a boundary region A are reconstructible in $\mathfrak{A}_A$ (entanglement-wedge reconstruction); (ii) the code reproduces the quantum-corrected (Faulkner– Lewkowycz–Maldacena / quantum-extremal-surface) form of the Ryu–Takayanagi relation $S(A) = \dfrac{|\gamma_A|}{4G_N} + S_{\text{bulk}}$, with γ_A the minimal bulk cut homologous to A and S_{bulk} the bulk entanglement correction (the classical RT law is the $S_{\text{bulk}} \to 0$ leading term); (iii) it exhibits negative tripartite information, the signature of the holographic entanglement structure [17, 13, 15, 14].*

Remark 8.4 (The RT surface is a homology representative). [EST]/[HEU]. The "homologous to A" clause in the Ryu–Takayanagi / quantum-extremal-surface formula is literally a statement in relative homology: γ_A is a representative of the class in $H_{d-1}(\text{bulk}, A)$ cut out by A, and the minimality (extremality, with the FLM bulk-entropy correction included) selects a geometric representative of a topological class [12, 14]. This is a second, independent appearance — alongside the surface code (Theorem 5.2) — of "cohomological/homological data controls where information lives." In the surface code the homology is of the code's own surface; in RT it is of the emergent bulk. The [HEU] step is to read these two as instances of one condensed-cohomological pattern; the [EST] content is each toy model on its own.

8.3 Reconstruction as descent

Proposition 8.5 (Reconstruction sheaf of a code)**.** [EST] *Fix a code V with a family of correctable regions closed under the relevant unions. The assignment*

$$A \longmapsto \{\, \textit{logical operators reconstructible in the region algebra } \mathfrak{A}_A \,\} \tag{6}$$

is a presheaf of algebras whose sections over the full boundary are the logical operators, and whose descent along a cover by correctable regions is governed exactly by the code's recovery structure: a compatible family of local reconstructions glues to a global logical operator iff the regions' recovery channels agree on overlaps. When they do not (e.g. across an entanglement-wedge transition / Ryu–Takayanagi phase transition), the obstruction is a genuine higher class.

Proof sketch. Reconstructibility is monotone under enlarging A within the correctable family, giving a presheaf; compatibility on overlaps is agreement of the two recovery channels, i.e. the 0-cocycle condition; a phase transition in the minimal surface changes which wedge contains the bulk point, so the two local reconstructions cease to agree and the gluing fails — a nonzero $\check{\mathrm{H}}^1$ of the reconstruction presheaf [16, 15]. Full formalization is left to Part IV/VI. □

9 Geometry from Compatibility

We can now state the paper's second, more ambitious speculative hypothesis, and immediately quarantine it with the status calculus.

9.1 The hypothesis

[SPEC] Geometry as the moduli of informational compatibility

Hypothesis 9.1 (Geometry as informational compatibility)**.** [SPEC] Suppose Conjecture 7.3 holds, so that a global informational state defines, on every cover of every context, a class in a condensed (co)homology theory. We hypothesize that a *geometry* — a condensed/topological, and ultimately Lorentzian, structure on the space of contexts — is not additional data but is *determined* by the totality of these compatibility classes: two contexts are "near" when their local states glue with small obstruction, and the emergent metric/causal structure is a functional of the class data $\{[\,c_\omega(\mathfrak{U})\,]\}$. Formally, geometry is proposed to be an invariant of the object $R\Gamma(-, \mathcal{O}_\omega) \in \mathcal{D}(\mathrm{Cond}(\mathrm{Vect}))$, in the same aspirational sense that Connes's spectral distance is an invariant of a spectral triple and Ryu–Takayanagi ties bulk distance to boundary entanglement.

9.2 Status accounting

Where this sits in the status calculus

The hypothesis of Hypothesis 9.1 is the condensed-cohomological upgrade of Van Raamsdonk's "entanglement builds spacetime" and ER = EPR [19, 18]. Both are [HEU]/[SPEC] even within AdS/CFT and are *not* established as a general mechanism outside holographic duality (KB §7.3). Composing the ([SPEC]) Conjecture 7.3 with a ([SPEC]) geometry-from-cohomology map yields, by worst-component-wins, a [SPEC] composite — and, as the series' composition discipline notes, stating a speculation twice does not raise its status. We record this rather than disguise it. The honest deliverable of Hypothesis 9.1 is a *research target* for Parts V–VI, phrased so that it could in principle be refuted (e.g. by exhibiting two states with identical cocycle data but forced-distinct emergent geometries), not a result.

Principle 9.2 (Information-first reading of the program)**.** [HEU] The three inversions of Parts I–III compose into a single methodological stance: *condensed substrate before manifold* (Part I), *observable before geometry* (Part II), *information before observable* (Part III). Read forward, the arrow is

$$\text{informational data} \xrightarrow{\text{gluing}} \text{cohomological obstructions} \xrightarrow{\text{moduli}} \text{geometry}.$$

The first arrow is [EST] (this paper's cohomology), the reading of its classes as information is [HEU], and the second arrow is [SPEC] (Hypothesis 9.1). Part III's contribution is to make the *first* arrow rigorous and to state the *second* precisely enough to be attacked.

10 Results and Formalization

10.1 Summary of results

Theorem 10.1 (Consolidated statement of Part III)**.** *The following hold with the indicated status.*

(1) [EST] *Propositions 3.2 and 3.4: Čech/derived cohomology is the exact obstruction theory of gluing; the connecting map δ is the universal gluing obstruction, and on the condensed site it is computed by Stonean hypercovers (Proposition 3.5).*

(2) [EST] *Definition 2.5 and Remark 2.4: the derived category of condensed (solid/liquid) vector spaces is the well-behaved home for state-valued obstruction classes, where naive topological vector spaces fail to be abelian.*

(3) [EST] *Theorem 5.2: the toric/surface code realizes "cohomology = protected information" as a theorem, with logical operators $\cong H_1(\Sigma; \mathbb{Z}/2)$ and distance a systole.*

(4) [EST] *Proposition 6.1 and Theorems 6.2 to 6.4: the state presheaf fails descent; the marginal problem, monogamy, and strong subadditivity are the rigorous guardrails on any cohomological account, with SSA-equality states = Markov-reconstructible = "locally trivial."*

(5) [EST] *Propositions 8.2 and 8.5 and Theorem 8.3: complementary recovery, the HaPPY code, and Ryu–Takayanagi realize state reconstruction as descent for a reconstruction presheaf, within their toy-model regimes.*

(6) [SPEC] *Conjecture 7.3: entanglement across a cover is proposed to be the nonvanishing of a condensed cosheaf (factorization) homology class of the state data on unions, with separable/Markov states the chain-trivial ones — a conjecture, contrasted explicitly with (1)–(5); the dual sheaf obstruction on overlaps governs the marginal problem.*

(7) [SPEC] *Hypothesis 9.1: geometry is proposed to be the moduli of informational compatibility, inheriting the* [SPEC] *status of the ER = EPR lineage.*

10.2 Haskell formalization

The accompanying Haskell package (four modules in the Part III `src` directory, compiled under `-Wall` with no warnings) makes the [EST] core executable. It is a finite, decidable shadow of the theory — ordinary $\mathbb{Z}/2$ and rational linear algebra standing in for the condensed coefficients — chosen so that every claim it checks is genuinely [EST].

- `Cech.hs` builds the Čech complex of a finite cover with $\mathbb{Z}/2$ coefficients, computes d^0, d^1, and returns $\dim \check{\mathrm{H}}^0$ and $\dim \check{\mathrm{H}}^1$ by Gaussian elimination over $\mathbb{Z}/2$; on the two-arc cover of a loop it returns $\check{\mathrm{H}}^1 = \mathbb{Z}/2$ (Proposition 3.2, Theorem 5.2 in miniature).
- `Marginal.hs` represents small density matrices, computes partial traces (marginals), and checks marginal compatibility of a candidate global state against a system of local states — exhibiting the Bell state and the product of its marginals as distinct states with identical marginals (Proposition 6.1, Proposition 4.2).
- `Stabilizer.hs` implements a small stabilizer code (the $[[4, 2, 2]]$ / smallest surface-code patch), computes its logical operators as the quotient (centralizer of the stabilizer)/(stabilizer), and demonstrates single-erasure correction (Definition 8.1, Proposition 8.2, Theorem 5.2).
- `Main.hs` runs all three demonstrations with commentary.

The code computes *only* [EST] quantities: cohomology of an explicit finite cover, marginal (in)compatibility, and stabilizer logical operators. It deliberately does *not* attempt to compute the speculative c_ω of Definition 7.1, since no algorithm for it is claimed.

11 Discussion

11.1 Explicit non-identifications

Per the discipline of the series (KB §7.3), we list what must not be silently merged.

- *Cohomology of the surface code* (Theorem 5.2, ordinary cohomology of a genuine surface, [EST]) is *not* the conjectural *condensed* cosheaf homology of an entangled state (Conjecture 7.3, [SPEC]). The former is a theorem about a space; the latter proposes to dispense with the space.

- *Contextuality cohomology* (Abramsky–Brandenburger, [EST]) is *not* the same as the proposed entanglement cohomology (Remark 7.4): different coefficients (empirical models vs. condensed states), different target (nonlocality of outcomes vs. entanglement of the state).

- *Entanglement entropy* (a real-valued invariant) is *not* claimed to be a cohomological dimension (Section 7.3); at most a rank/filtration of the cocycle is.

- *Ryu–Takayanagi / ER = EPR* ([HEU]/[SPEC], native to AdS/CFT) is *not* an established general mechanism, and Hypothesis 9.1 inherits, not improves, that status.

- *Reconstruction as descent* (Proposition 8.5) holds for the toy codes; it is *not* claimed for full quantum gravity.

11.2 Limitations and status composition

The mathematical substrate of this paper is genuinely [EST]: cohomology, derived categories, condensed vector spaces, the marginal problem, and the holographic toy models are all rigorous within their stated hypotheses. The two headline hypotheses are [SPEC] and stay [SPEC] under composition. The most important limitation is honest to name: we do *not* construct the cohomology theory of Conjecture 7.3; we specify the properties it must have and show they are consistent with the established guardrails. Making $\check{\mathrm{H}}^{\bullet}_{\mathrm{Cond}}(-, \mathcal{O}_\omega)$ an actual functor with target $\mathcal{D}(\mathrm{Cond}(\mathrm{Vect}))$ — in particular defining $\star_{ij}$ globally and proving the vanishing criterion (A) — is the central open problem this paper hands forward. A second limitation: the identification of a class-level invariant with an LOCC monotone (part (B)) is asserted, not proved, and the QMA-completeness of the marginal problem (Theorem 6.2) means any such theory is at least as hard as its guardrails.

11.3 What Part III hands to Parts IV–VII

Part III hands forward three interfaces. To **Part IV** (fields without background manifolds): the state complex $R\Gamma(\mathfrak{U}, \mathcal{O}_\omega)$ valued in solid/liquid modules is the kinematic object a condensed field theory must produce as its space of states, and Remark 2.4 is the reason the field values belong in solid/liquid modules. To **Part V** (geometry as a realization functor): Hypothesis 9.1 is the informational input to the reconstruction functor — the cocycle data $\{[\, c_\omega(\mathfrak{U})\,]\}$ is what a condensed-Tannakian reconstruction would take as its fibre-functor data. To **Part VI** (representation-theoretic quantum gravity): Conjecture 7.3 composed with Hypothesis 9.1 is exactly the "gravity from gluing of informational structures" arrow, correctly labelled [SPEC] and correctly bounded by the guardrails of Section 6.2.

Claim	Status	Where
Čech/derived cohomology = gluing obstruction	[EST]	Propositions 3.2 and 3.4
Condensed vector spaces as coefficient home	[EST]	Definition 2.5 and Remark 2.4
Surface code: logicals $\cong H_1(\Sigma; \mathbb{Z}/2)$	[EST]	Theorem 5.2
Marginal problem / monogamy / SSA	[EST]	Theorems 6.2 to 6.4
Complementary recovery, HaPPY, RT/FLM	[EST](toy models)	Proposition 8.2 and Theorem 8.3
$H^{\geq 1}$ read as protected information	[HEU]	Remarks 4.4 and 5.3
Entanglement = condensed cosheaf class	[SPEC]	Conjecture 7.3
Geometry = moduli of compatibility	[SPEC]	Hypothesis 9.1

Table 1: Established/heuristic/speculative census for Part III. The [SPEC] rows are the paper's original proposals; every row above them is rigorous within its stated hypotheses.

12 Conclusion

Part III sits on the seam of the program and is written to be trustworthy there. Its established core is a single idea developed carefully: the failure of local informational data to glue is measured by cohomology, and the natural home for the obstruction classes — once the coefficients are quantum states — is the derived category of condensed vector spaces, precisely because that is the smallest enlargement of "topological vector space" in which the required homological algebra exists. That core is illustrated by two rigorous worked examples where "cohomology equals protected information" is a theorem: the toric/surface code and, homologically, the Ryu–Takayanagi surface. Against this established background we have stated, boxed and labelled, two speculative hypotheses — that entanglement *is* a condensed Čech class, and that geometry is the moduli of informational compatibility — with an explicit account of what would be required to prove them, what constraints (marginal problem, monogamy, strong subadditivity) they must respect, and why, under the series' worst-component-wins discipline, their composite status is and remains [SPEC]. The honest deliverable is not a theorem about emergent geometry but a precisely posed obstruction-theoretic research program, handed to Parts IV–VII with its epistemic status intact.

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