Fields Without Background Manifolds

Part IV of *Toward a Condensed Representation Theory of Physics*

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Abstract

A classical field is a map $\varphi\colon M \to \mathbb{R}$ from a smooth manifold to a coefficient space; its configuration space $C^\infty(M)$ is a topological vector space, and its quantum theory is built on spaces of test functions and distributions. This paper asks whether the manifold M can be demoted from a presupposed background to a derived datum, by replacing $\varphi\colon M \to \mathbb{R}$ with a condensed-vector-space–valued object $\varphi\colon C \to \mathrm{Cond(Vect)}$ over a condensed base C. We develop the mathematics that makes this substitution well-posed and mark, throughout, the boundary between what is established and what is a research proposal. *Established mathematics* [EST]: the category of topological vector spaces is not abelian, whereas condensed abelian groups are a Grothendieck abelian category (Clausen–Scholze); solid modules and liquid vector spaces are the full subcategories that repair the pathologies of "complete topological modules" and "complete topological vector spaces," carrying well-behaved completed tensor products and a six-functor formalism through the theory of analytic rings; classical nuclear spaces of test functions and distributions embed faithfully, with the condensed internal tensor and Hom recovering their completed topological counterparts and rendering the Schwartz kernel theorem an adjunction; and the free scalar field over a fixed globally hyperbolic spacetime – its symplectic space of solutions, its CCR/Weyl $*$-algebra, its Pauli–Jordan propagator, its quasi-free vacuum two-point function and GNS data – is faithfully re-encoded as a $*$-algebra object and condensed functionals in Cond(Vect). *Heuristic dictionary* [HEU]: that solid/liquid distribution theory is the right home for the singular kernels of interacting quantum field theory. *Speculative hypothesis* [SPEC], clearly boxed: that a full quantum field theory – ultimately including interactions – can be defined with fields valued in condensed modules over an analytic ring on an *abstract* condensed base carrying no presupposed smooth manifold, with the manifold recovered, when it exists, as a realization. We give the free-field construction as the honest test case, isolate precisely which of its ingredients are already manifold-free and which still secretly invoke M, and state the open problems that separate the two.

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1 Introduction

1.1 The modular program and its governing perspective

This is Part IV of a modular, hierarchical seven-part program organized by a single perspective:

> *Physics is the study of realizations of condensed mathematical structures.*

The program is *modular*, not unified: each Part isolates one interface between condensed mathematics and physics, states precisely what it establishes, and composes with its predecessors only through explicitly declared hand-offs. Part I [1] replaced the smooth manifold by the condensed set as the kinematic substrate, exhibiting a fully faithful embedding $\mathrm{Top}_{\mathrm{cg}} \hookrightarrow \mathrm{Cond}(\mathrm{Set})$ under which "manifold" becomes a special, faithfully retained case of "condensed space." Part II [2] placed *observables* before geometry, building a condensed sheaf of observable algebras on the pro-étale site and proving a descent theorem for the algebra side. Part III [3] took *information* as primitive, measuring the failure of the *state* side to satisfy descent by a class in the derived category of condensed abelian (respectively, solid or liquid) coefficients. The present Part IV takes the next concrete object – the *field* – and asks what becomes of it when the manifold underneath is no longer presupposed.

1.2 From $\varphi\colon M \to \mathbb{R}$ to $\varphi\colon C \to \mathrm{Cond}(\mathrm{Vect})$

A scalar field on a spacetime M is, in the simplest instance, a smooth map $\varphi\colon M \to \mathbb{R}$; more generally a section of a vector bundle $E \to M$. The space of such fields, $\mathfrak{F} = C^\infty(M)$ or $\Gamma(M, E)$, is a topological (typically Fréchet) vector space, and the entire apparatus of classical and quantum field theory – variational calculus, Green's functions, test functions, distributions, operator-valued distributions, states – is functional analysis performed on $\mathfrak{F}$ and its relatives. Two structural facts drive this paper.

First, the ambient category is wrong. Topological vector spaces do *not* form an abelian category: continuous linear bijections need not be isomorphisms, cokernels are computed with a quotient topology that discards information, and the completed tensor product is delicate and non-canonical (projective versus injective versus inductive). Homological algebra – the language in which locality, gluing, obstruction, and descent were phrased in Parts II and III – does not run natively on TVS. The Clausen–Scholze theory of condensed mathematics repairs exactly this: Cond(Ab) *is* a Grothendieck abelian category, and its full subcategories of *solid* modules and *liquid* vector spaces are the correct homes for "complete" algebra and analysis [5, 6, 7].

Second, the manifold M enters twice, and the two roles are separable. It is the *base* over which fields are defined, and it is the source of the *analytic structure* (smoothness, integration, propagators) that makes the field space a well-behaved functional-analytic object. The proposal of this paper is to move the analytic structure into the coefficient category – to make the *target* a condensed/solid/liquid module – so that the field becomes an object

$$\varphi\colon C \longrightarrow \mathrm{Cond}(\mathrm{Vect}),$$

a morphism into (or an internal section valued in) a condensed vector space over a condensed base C, with the base no longer required to be a manifold. Section 3 makes this precise. The honest status of the substitution is the subject of the whole paper: over a fixed manifold it is an established re-encoding; with the manifold removed it is a research program.

1.3 The seam this Part occupies

Part III was the hinge between the established first half of the series and its speculative second half. Part IV is the first Part on the far side of that hinge, and it is written to keep the seam visible *within a single construction*. The free scalar field is our test case precisely because every one of its ingredients can be examined individually for manifold-dependence. Its coefficient data – test functions, distributions, the CCR relations, the vacuum two-point function, the GNS Hilbert space – transports *losslessly* into Cond(Vect); this is [EST]. Its base data – the globally hyperbolic manifold that carries the Klein–Gordon operator and the Pauli–Jordan propagator – does *not* yet transport, because we still write those operators using the smooth structure of M; asserting that it can be removed is [SPEC]. Naming which is which, sentence by sentence, is the methodological content of this Part.

1.4 The epistemic discipline: EST, HEU, SPEC

Following the series, every substantive assertion carries one of three labels.

- [EST] — *established*: rigorous, citable mathematics, or a rigorously proved physics-adjacent theorem within its stated hypotheses.
- [HEU] — *heuristic*: a physically motivated dictionary entry or interpretive reading, not itself a theorem.
- [SPEC] — *speculative*: an ontological or physical hypothesis of this program, with no established precedent.

The labels compose by a *worst-component-wins* law: a chain of reasoning is capped at the status of its weakest link [3, 20]. The mathematics of condensed, solid, and liquid modules and of the free-field algebra is [EST]; the reading of solid/liquid distribution theory as the right home for interacting kernels is [HEU]; and the headline claim – that a full, ultimately interacting quantum field theory can be defined manifold-freely – is [SPEC], and remains [SPEC] however much [EST] machinery is marshalled beneath it. We restate this in Section 8 and tabulate it in Table 2.

1.5 What this Part is not

Three explicit non-identifications, maintained per the discipline of the series [1, 3]:

1. Condensed mathematics (Clausen–Scholze; external/analytic, sheaves on profinite sets) is *not* cohesive homotopy type theory (Schreiber; internal/synthetic, modal type theory) and is *not* topos quantum theory (Isham–Döring; presheaves on a context category of commutative subalgebras). All three deploy topos/sheaf technology; only the first is used here, and the field construction below is external and analytic, not synthetic [18, 19].

2. The condensed re-encoding of the free field over a fixed manifold is *not* a manifold-free construction of that field. We prove the former and conjecture the latter; the two are kept typographically distinct throughout Sections 5 and 7.

3. The claim "solid/liquid modules are the right functional analysis for QFT" is established for the *linear* (free, Gaussian) theory and is [HEU] for the interacting theory; we do not smuggle the rigor of the former into the latter.

1.6 Contributions and outline

By register, the contributions are:

[EST] Section 2: a self-contained account of why TVS fails to be abelian and how Cond(Ab), solid modules, and liquid vector spaces repair it, with the completed tensor products and the analytic-ring/six-functor packaging stated precisely and cited.

[EST] Section 3: a precise definition of a field over a condensed base as an internal morphism / section valued in a condensed module (Definition 3.3), with a diagram (Figure 1) exhibiting the manifold picture as a faithful special case.

[EST] Section 4: the condensed reformulation of Schwartz distribution theory – test functions and distributions as condensed vector spaces, the pairing as an internal-Hom evaluation, and the Schwartz kernel theorem as a tensor–Hom adjunction (Theorem 4.4).

[EST] Section 5: the free scalar CCR/Weyl algebra, Pauli–Jordan propagator, quasi-free vacuum, and GNS data over a fixed globally hyperbolic M, re-encoded as a $*$-algebra object and condensed functionals in Cond(Vect) (Theorems 5.4 and 5.5), together with an explicit finite-mode shadow (Section 5.3) realized in the accompanying code.

[EST]/[HEU] Section 6: quantum states and operator algebras condensed-ly – condensed nets, states as condensed functionals, and the status of Tomita–Takesaki modular theory in this language.

[SPEC] Section 7: the boxed research hypothesis (Hypothesis 7.1) that a full QFT can be defined manifold-freely, an itemized audit of which free-field ingredients are already manifold-free and which are not, and the precise open problems.

- Section 8: discussion, the EST/HEU/SPEC census (Table 2), limitations, and relations to adjacent programs; Section 9: conclusion.

2 Mathematical framework: condensed vector spaces

This section fixes notation and recalls the established mathematics on which the paper rests. Nothing here is new; the aim is to state, precisely and citably, the sense in which condensed mathematics supplies an abelian/stable home for functional analysis. Parts I–III are cited rather than reproved [1, 2, 3, 5].

2.1 Condensed sets and condensed modules

Definition 2.1 (Pro-étale site of a point; condensed sets [EST])**.** Let ProFin be the category of profinite sets (cofiltered limits of finite discrete sets), equipped with the Grothendieck topology whose covers are finite jointly surjective families. A *condensed set* is a sheaf $T\colon \mathrm{ProFin}^{\mathrm{op}} \to \mathrm{Set}$ for this topology that is additionally small (accessible); equivalently $T(\emptyset) = *$, $T(S \sqcup S') = T(S) \times T(S')$, and for every surjection $S' \twoheadrightarrow S$ the sequence $T(S) \to T(S') \rightrightarrows T(S' \times_S S')$ is an equalizer. The sheaf topos may be computed equivalently on profinite sets, on compact Hausdorff spaces, or on the extremally disconnected (Stonean) profinite sets, which form a basis [5]. A *condensed abelian group* (resp. *condensed R-module*) is an abelian-group (resp. R-module) object internal to condensed sets.

Proposition 2.2 (Comparison with topological spaces [EST], [5, Prop. 1.7])**.** *The functor $\mathrm{Top} \to \mathrm{Cond(Set)}$, $X \mapsto \big(S \mapsto \mathrm{Cont}(S, X)\big)$, is faithful, and fully faithful when restricted to compactly generated (weak Hausdorff) spaces. It carries topological abelian groups to condensed abelian groups and admits a left adjoint sending T to the set $T(*)$ with the quotient topology from $\coprod_{S \to T} S \twoheadrightarrow T(*)$.*

We write $\underline{X} \in \mathrm{Cond(Set)}$ for the condensed set attached to a topological space X, and similarly $\underline{V}$ for a topological abelian group or vector space V.

2.2 Why topological vector spaces are the wrong category

The technical motivation for the whole paper is a failure of the classical category, which we state sharply.

Proposition 2.3 (TVS and TopAb are not abelian [EST])**.** *The category of Hausdorff topological abelian groups (a fortiori topological vector spaces) is additive but not abelian. Concretely, the continuous identity map $\mathbb{R}_{\mathrm{disc}} \to \mathbb{R}$ from the reals with the discrete topology to the reals with the Euclidean topology is a monomorphism and an epimorphism (it is bijective and continuous) but not an isomorphism; hence there is a morphism that is mono and epi yet not invertible, and the canonical map from the coimage to the image is not an isomorphism. Equivalently, kernels and cokernels of continuous homomorphisms exist but the first isomorphism theorem fails at the level of topological groups.*

Discussion. Bijectivity and continuity of $\mathbb{R}_{\mathrm{disc}} \to \mathbb{R}$ are immediate; its inverse is not continuous, so it is not an isomorphism of topological groups. In an abelian category every morphism that is simultaneously mono and epi is an isomorphism; the displayed map witnesses that TopAb is not abelian. See [5] for the systematic discussion, where this example is the motivating pathology that condensed mathematics is designed to cure. $\square$

Proposition 2.4 (Cond(Ab) is abelian [EST], [5]). *The category* Cond(Ab) *of condensed abelian groups is a Grothendieck abelian category: it has all small limits and colimits, a generator, filtered colimits are exact, and it has enough projectives (the free condensed abelian groups* $\mathbb{Z}[S]$ *on extremally disconnected* S*). The embedding* $\mathrm{TopAb}_{\mathrm{cg}} \hookrightarrow \mathrm{Cond(Ab)}$ *of Proposition 2.2 is fully faithful and exact-reflecting in the sense that a short exact sequence of condensed abelian groups whose terms are topological need not be a topological short exact sequence – the condensed category is strictly larger and better behaved. Consequently the derived category* D(Cond(Ab)) *is a well-behaved stable* ∞*-category in the sense of [11], and all of homological algebra is available.*

Propositions 2.3 and 2.4 are the entire justification for the move $\varphi\colon M \to \mathbb{R} \rightsquigarrow \varphi\colon C \to \mathrm{Cond(Vect)}$: the target category on the right supports the homological algebra that the target category on the left does not.

2.3 Solid modules

Working over a discrete ring R (for us $R = \mathbb{Z}$ or $R = \mathbb{C}$ discrete), the naive free condensed module $R[S]$ on a profinite $S = \varprojlim_i S_i$ is not complete; its completion is the module of R-valued measures.

Definition 2.5 (Solid module [EST], [5]). For a profinite set $S = \varprojlim_i S_i$ set $R[S]^{\blacksquare} := \varprojlim_i R[S_i]$ (the free *solid* module, i.e. the module of R-measures on S). A condensed R-module M is *solid* if for every profinite S the natural map induces a bijection

$$\mathrm{Hom}_{\mathrm{Cond}(R)}\big(R[S]^{\blacksquare}, M\big) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Cond}(R)}\big(R[S], M\big).$$

Informally: every "profinitely indexed null-summable" family in M has a well-defined sum.

Theorem 2.6 (Structure of solid modules [EST], [5, 10]). *The solid R-modules* Solid_R *form a full abelian subcategory of* $\mathrm{Cond}(R)$*, closed under all limits, colimits, and extensions, with a left adjoint* solidification $M \mapsto M^{\blacksquare}$ *to the inclusion. There is a unique symmetric monoidal completed tensor product* $\otimes_{\blacksquare}$ *on* Solid_R *making solidification monoidal, with unit R and internal Hom, and the inclusion* $\mathrm{Solid}_R \hookrightarrow \mathrm{Cond}(R)$ *preserves the relevant* Ext *and* Tor*. Profinite R-modules embed so as to preserve* Ext *and* Tor*; Nöbeling's theorem finds its natural home here (formally verified via the solid lens [10]).*

Example 2.7 ([EST]). $\mathbb{Z}_p = \varprojlim_n \mathbb{Z}/p^n$ is solid, and $\mathbb{Z}_p \otimes_{\blacksquare} \mathbb{Z}_p = \mathbb{Z}_p$; more generally the solid tensor product of two profinite modules is their completed tensor product in the classical sense. The discrete $\mathbb{R}$ is *not* solid over $\mathbb{Z}$ in a way that repairs archimedean analysis – for that one needs the liquid theory of Section 2.4, because $\mathbb{R}$ carries no profinite integral structure.

2.4 Liquid vector spaces and analytic rings

For archimedean coefficients ($\mathbb{R}$ or $\mathbb{C}$) solidity is replaced by a controlled-growth condition, producing the *liquid* theory – the functional-analytic core of this paper.

Definition 2.8 (p-liquid vector space [EST], [6, 7]). Fix a real parameter $0 < p \leq 1$. The category $\mathrm{Liq}_p(\mathbb{R})$ of *p-liquid* $\mathbb{R}$-vector spaces is a full subcategory of condensed $\mathbb{R}$-vector spaces, defined by a summability/growth condition on measures with values indexed over profinite sets (the archimedean analogue of Definition 2.5, using the "p-measures" functor $\mathcal{M}_p(-)$ in place of $R[-]^{\blacksquare}$). It is closed under kernels, cokernels, and extensions inside condensed $\mathbb{R}$-vector spaces, and it possesses all small limits and colimits; limits and finite colimits agree with those of bare condensed $\mathbb{R}$-vector spaces, whereas infinite colimits are computed by applying the left-adjoint liquification to the condensed colimit (so an infinite direct sum must be liquified).

Theorem 2.9 (Liquid vector spaces are abelian and homologically well-behaved [EST], [6, 7, 9]). *For each $0 < p \leq 1$, $\mathrm{Liq}_p(\mathbb{R})$ is an abelian category closed under extensions in* $\mathrm{Cond}(\mathbb{R}\text{-Vect})$*, carrying a symmetric monoidal completed tensor product $\otimes_{\mathrm{L}}$ with unit $\mathbb{R}$ and internal Hom $\underline{\mathrm{Hom}}(-,-)$. The key technical input – that the relevant higher* Ext*-groups against spaces of measures vanish, so that liquid modules form a well-behaved (derived) tensor category – is the content of Clausen–Scholze's "Analytic Geometry" Theorem 9.1, whose most delicate step was the object of the* Liquid Tensor Experiment *and has been fully formally verified in Lean [9]. In particular* $\mathrm{D}(\mathrm{Liq}_p(\mathbb{R}))$ *is a well-behaved stable ∞-category and $\otimes_{\mathrm{L}}$ has a derived functor with the expected base-change properties.*

Definition 2.10 (Analytic ring and its six-functor formalism [EST], [7, 8]). An *analytic ring* is a pair $\mathcal{A} = (\underline{\mathcal{A}}, \mathrm{D}(\mathcal{A}))$ where $\underline{\mathcal{A}}$ is a condensed (commutative, or E_∞-) ring and $\mathrm{D}(\mathcal{A})$ is a full stable subcategory of condensed $\underline{\mathcal{A}}$-modules – "the complete modules" – closed under the relevant limits, colimits, and extensions and compatible with base change. The pair $(\mathbb{Z}, \mathrm{Solid}_{\mathbb{Z}})$ is the solid analytic ring; $(\mathbb{R}, \mathrm{Liq}_p(\mathbb{R}))$ is the liquid analytic ring. Analytic rings support a full six-functor formalism $(f^*, f_*, \otimes_{\mathrm{L}}, \underline{\mathrm{Hom}}, f_!, f^!)$ with base change and projection formulas [8]; this is the established framework for a general analytic geometry encompassing the complex-analytic and rigid-analytic cases, and it is the machinery on which a manifold-free field theory would have to be built.

Remark 2.11 (Why this matters here [EST]/[HEU]). Theorems 2.6 and 2.9 give exactly what Proposition 2.3 denied: a category of "complete" modules that is abelian/stable, with a canonical, associative, unital completed tensor product and an internal Hom, closed under the operations (kernels, cokernels, extensions, derived tensor, derived Hom) that a Lagrangian, path-integral, or operator-algebraic field theory manipulates. That the linear apparatus of field theory lives natively here is [EST]; that this suffices for the *nonlinear* (interacting) theory is [HEU], and we mark it so throughout.

3 Fields as condensed-vector-space–valued objects

We now make the substitution of Section 1.2 precise. The guiding principle is that a field should be an object of an abelian/stable category from the outset, so that the local-to-global language of Parts II–III applies to it directly.

3.1 The classical datum

Definition 3.1 (Classical field configuration [EST])**.** Let M be a smooth manifold and $E \to M$ a smooth (finite-rank real) vector bundle. A *classical field configuration* is a section $\varphi \in \Gamma(M, E)$; for $E = M \times \mathbb{R}$ this is a smooth function $\varphi\colon M \to \mathbb{R}$. The *configuration space* is $\mathfrak{F} = \Gamma(M, E)$, a real Fréchet space under the C^∞ topology (compact-uniform convergence of all derivatives).

By Proposition 2.3, $\mathfrak{F}$ is not an object of an abelian category. Its condensed avatar $\underline{\mathfrak{F}}$ is, and is moreover liquid (nuclear); this is the first payoff.

Proposition 3.2 (Nuclear field spaces are liquid [EST], [6, 7])**.** *For M second countable, $\Gamma(M, E)$ with the C^∞ topology is a nuclear Fréchet space; its condensed avatar $\underline{\Gamma(M, E)}$ lies in $\mathrm{Liq}_p(\mathbb{R})$ for the relevant p, and the functor $V \mapsto \underline{V}$ from nuclear Fréchet spaces to liquid vector spaces is fully faithful and carries the completed projective/injective topological tensor product (which coincide, by nuclearity) to the liquid tensor product $\otimes_{\mathrm{L}}$.*

3.2 The condensed field

Definition 3.3 (Field over a condensed base [EST])**.** Let $\mathcal{A}$ be an analytic ring (e.g. the liquid ring $(\mathbb{R}, \mathrm{Liq}_p(\mathbb{R}))$), let $V \in \mathrm{D}(\mathcal{A})$ be a coefficient module ("the values"), and let $C \in \mathrm{Cond(Set)}$ be a condensed base object – more generally a condensed anima. An *$\mathcal{A}$-valued field on C with values in V* is a morphism of condensed sets

$$\varphi\colon C \longrightarrow U(V),$$

where $U\colon \mathrm{D}(\mathcal{A}) \to \mathrm{Cond(Set)}$ is the forgetful functor to the underlying condensed set; equivalently, a global section of the constant condensed sheaf $\underline{V}$ over C. The *field configuration space* is the internal mapping object

$$\mathfrak{F}(C, V) \;:=\; \underline{\mathrm{Hom}}_{\mathcal{A}}\big(\mathcal{A}[C], V\big) \;\in\; \mathrm{D}(\mathcal{A}),$$

the module of $\mathcal{A}$-linear maps out of the free $\mathcal{A}$-module $\mathcal{A}[C]$ on C; it is itself an object of the abelian/stable category $\mathrm{D}(\mathcal{A})$.

The content of Definition 3.3 is that $\mathfrak{F}(C, V)$ is a condensed module – an object of an abelian/stable category – *regardless of whether C is a manifold*. The base C enters only through the free module $\mathcal{A}[C]$; the analytic structure lives in $\mathcal{A}$ and V. This is the formal separation of "base" from "analytic structure" promised in Section 1.2.

Proposition 3.4 (The classical field is a faithful special case [EST])**.** *Taking $C = \underline{M}$, $\mathcal{A} = (\mathbb{R}, \mathrm{Liq}_p(\mathbb{R}))$, and $V = \underline{\mathbb{R}}$ in Definition 3.3, the set of condensed fields $\varphi\colon \underline{M} \to U(\underline{\mathbb{R}})$ is in natural bijection with the set of continuous maps $M \to \mathbb{R}$, and the configuration space $\mathfrak{F}(\underline{M}, \underline{\mathbb{R}})$ contains $\underline{C^\infty(M)}$ as the liquid subobject of smooth fields. In particular the classical field theory is faithfully embedded, and Figure 1 commutes.*

Sketch. Full faithfulness of $\underline{(-)}$ on compactly generated spaces (Proposition 2.2) gives the bijection on maps: $\mathrm{Hom}_{\mathrm{Cond}}(\underline{M}, \underline{\mathbb{R}}) = \mathrm{Cont}(M, \mathbb{R})$. Smoothness is an extra condition cut out inside the liquid vector space $\underline{C^\infty(M)} \subset \underline{C^0(M)}$ by the (condensed) differential operators, all of which are morphisms of liquid vector spaces by Proposition 3.2. Hence $C^\infty(M)$ appears as a liquid subobject, and no classical datum is lost. ◻

$$
\begin{tikzcd}
M \arrow[r, "\varphi"] \arrow[d, "\underline{(-)}"'] & \mathbb{R} \arrow[d, "\underline{(-)}"] \\
\underline{M} \arrow[r, "\underline{\varphi}"'] \arrow[d, equal] & \underline{\mathbb{R}} \arrow[d, equal] \\
C \arrow[r, "\varphi"'] & U(V)
\end{tikzcd}
$$

Figure 1: The move $\varphi\colon M \to \mathbb{R} \;\rightsquigarrow\; \varphi\colon C \to U(V)$. Top row: the classical field, a smooth map of a manifold to the coefficient line. Middle row: its image under the fully faithful embedding $\underline{(-)}\colon \mathrm{Top}_{\mathrm{cg}} \hookrightarrow \mathrm{Cond}(\mathrm{Set})$ (Proposition 2.2); no information is lost. Bottom row: the general condensed field of Definition 3.3, with $C = \underline{M}$ and $V = \underline{\mathbb{R}}$ recovering the classical case, but now C and V are free to be an abstract condensed base and an abstract analytic-ring module. The vertical equalities record that the classical field is the special case $C = \underline{M}$, $V = \underline{\mathbb{R}}$; removing that specialization is the content of Section 7.

Remark 3.5 (Sections, sheaves, and the Part II hand-off [EST]). Definition 3.3 is the pointwise (global-sections) shadow of a sheaf-theoretic object. In the language of Part II [2], a field is a global section of a condensed sheaf $\underline{V}$ on the pro-étale site of the base; the assignment $U \mapsto \mathfrak{F}(U, V)$ of local configurations to sub-objects $U \subseteq C$ is a sheaf of liquid vector spaces, and its descent is exactly the descent theorem of Part II applied to the coefficient $\underline{V}$. Thus "field" in this Part and "observable" in Part II are two sections of the same condensed-sheaf machine: the field is the kinematic section, the observable its linear dual.

4 Distribution theory in the condensed and solid setting

Quantum field theory is distribution theory: fields are operator-valued distributions, propagators are distributional kernels, and the singular structure of products of distributions is the entire content of renormalization. This section recasts the Schwartz picture in condensed terms and states precisely what is gained.

4.1 The classical Schwartz picture

Definition 4.1 (Test functions and distributions [EST]). For M a smooth manifold let $\mathcal{D}(M) = C_c^\infty(M)$ be the space of compactly supported smooth test functions with its usual (LF-)topology, and $\mathcal{E}(M) = C^\infty(M)$ the smooth functions with the Fréchet C^∞ topology. The *distributions* are the topological dual $\mathcal{D}'(M) = \mathcal{D}(M)'$ and the *compactly supported distributions* $\mathcal{E}'(M) = \mathcal{E}(M)'$. The canonical pairing $\langle -, - \rangle\colon \mathcal{D}'(M) \times \mathcal{D}(M) \to \mathbb{R}$ is separately continuous. All these spaces are nuclear.

The nuclearity in Definition 4.1 is the classical fact that makes the duality theory work, and it is precisely what liquid vector spaces internalize.

4.2 The condensed reformulation

Definition 4.2 (Condensed test functions and distributions [EST]). Let $\mathcal{D}_C := \underline{\mathcal{D}(M)}$ and $\mathcal{E}_C := \underline{\mathcal{E}(M)}$ be the liquid vector spaces attached (Proposition 3.2) to the classical test-function and smooth-function spaces. Define the *condensed distributions* as the internal dual

$$\mathcal{D}'_C := \underline{\mathrm{Hom}}_{\mathcal{A}}(\mathcal{D}_C, \underline{\mathbb{R}}), \qquad \mathcal{E}'_C := \underline{\mathrm{Hom}}_{\mathcal{A}}(\mathcal{E}_C, \underline{\mathbb{R}}),$$

computed in liquid $\mathbb{R}$-vector spaces. The pairing is the evaluation morphism $\mathrm{ev}\colon \mathcal{D}'_C \otimes_{\mathrm{L}} \mathcal{D}_C \to \underline{\mathbb{R}}$ in $\mathrm{Liq}_p(\mathbb{R})$.

Proposition 4.3 (The condensed dual recovers the Schwartz dual [EST], [6, 7]). *For M second countable, the internal dual $\mathcal{D}'_C$ of Definition 4.2 is the condensed avatar $\underline{\mathcal{D}'(M)}$ of the classical distribution space, and the evaluation morphism recovers the classical separately continuous pairing. Because $\mathcal{D}_C$ is nuclear, the natural map to the double dual $\mathcal{D}_C \to \underline{\mathrm{Hom}}_{\mathcal{A}}(\mathcal{D}'_C, \underline{\mathbb{R}})$ is an isomorphism of liquid vector spaces (reflexivity holds internally), and the internal-Hom bifunctor is exact in each variable in the derived sense.*

The gain over the classical picture is Proposition 4.3's last two clauses: reflexivity and the exactness of internal Hom are *clean* statements in $\mathrm{Liq}_p(\mathbb{R})$, whereas in TVS one must fight the failure of the category to be abelian (Proposition 2.3) at every step.

Theorem 4.4 (Schwartz kernel theorem as a tensor–Hom adjunction [EST], [6, 7]). *Let M, N be second countable manifolds and $\mathcal{D}_C(M), \mathcal{D}_C(N)$ the associated liquid test-function spaces. Then there are natural isomorphisms of liquid vector spaces*

$$\mathcal{D}'_C(M \times N) \;\cong\; \underline{\mathrm{Hom}}_{\mathcal{A}}\big(\mathcal{D}_C(M),\, \mathcal{D}'_C(N)\big) \;\cong\; \mathcal{D}'_C(M) \otimes_{\mathrm{L}} \mathcal{D}'_C(N).$$

Equivalently, the Schwartz kernel theorem – "every separately continuous bilinear form on $\mathcal{D}(M) \times \mathcal{D}(N)$ is an integral kernel in $\mathcal{D}'(M \times N)$" – is the tensor–Hom adjunction for the monoidal structure $\otimes_{\mathrm{L}}$ on nuclear liquid vector spaces.

Sketch. By Proposition 3.2 the functor $\underline{(-)}$ from nuclear spaces to liquid vector spaces is monoidal for $(\hat{\otimes}, \otimes_{\mathrm{L}})$; the classical Schwartz kernel theorem states precisely $\mathcal{D}'(M \times N) \cong \mathcal{D}(M)' \hat{\otimes} \mathcal{D}(N)' \cong L(\mathcal{D}(M), \mathcal{D}'(N))$ for nuclear $\mathcal{D}$. Applying the fully faithful monoidal $\underline{(-)}$ and the internal-Hom/tensor adjunction of Theorem 2.9 transports each isomorphism into $\mathrm{Liq}_p(\mathbb{R})$. Nuclearity guarantees the several classical tensor products coincide, so the target is unambiguous. $\square$

Remark 4.5 (Solid microlocal coefficients [HEU]). The singular support and wavefront-set refinements that control products of distributions (Hörmander; the Radzikowski microlocal spectrum condition for Hadamard states) are not yet expressed condensed-ly. The [HEU] expectation is that the six-functor formalism of analytic rings (Definition 2.10) – in particular the interplay of $f_!$ and $f^!$ over the diagonal – is the natural setting for a condensed microlocal analysis, and hence for the operator-product structure of interacting fields. We flag this as heuristic: no theorem to this effect exists, and Remark 2.11 applies.

5 Free fields, condensed-ly: the CCR construction

This is the paper's honest test case. We take the free scalar field over a *fixed* globally hyperbolic spacetime – the best-understood, fully rigorous quantum field theory – and show that its algebraic and state-theoretic data transports losslessly into Cond(Vect). We are scrupulous that the base M is still a manifold here; removing it is Section 7.

5.1 The classical free scalar field

Definition 5.1 (Free scalar phase space [EST], [12, 13, 14])**.** Let (M, g) be a globally hyperbolic Lorentzian manifold and $P = \Box_g + m^2$ the Klein–Gordon operator. There exist unique retarded and advanced Green's operators $E^\pm : \mathcal{D}(M) \to \mathcal{E}(M)$ with $PE^\pm = E^\pm P = \mathrm{id}$ and $\mathrm{supp}(E^\pm f) \subseteq J^\pm(\mathrm{supp}\, f)$, so that E^+ (future support) is retarded and E^- (past support) is advanced. The *causal propagator* (Pauli–Jordan function) is $E := E^+ - E^-$ (retarded minus advanced); its overall sign is fixed by the normalization $[\Phi(f), \Phi(h)] = i\, E(f, h)$ of Definition 5.2. The *phase space* is the real symplectic vector space

$$(V, \sigma), \qquad V := \mathcal{D}(M)_{\mathbb{R}} / \ker E \;\cong\; E(\mathcal{D}(M)_{\mathbb{R}}), \qquad \sigma([f], [h]) := \int_M f\,(Eh)\; \mathrm{dvol},$$

a well-defined antisymmetric nondegenerate form (the integral is the smearing of the Pauli–Jordan kernel). Here $\ker E = P\mathcal{D}(M)_{\mathbb{R}}$, so the quotient by $\ker E$ is exactly the imposition of the Klein–Gordon equation of motion $\Phi(Pf) = 0$, closing the loop with the operator P.

Definition 5.2 (Polynomial CCR $*$-algebra and its Weyl C^*-completion [EST], [12, 13])**.** We distinguish two related algebras. The *polynomial CCR $*$-algebra* $\mathfrak{A}_{\mathrm{CCR}}(V, \sigma)$ is the unital $*$-algebra generated by hermitian symbols $\Phi(f)$, $f \in \mathcal{D}(M)_{\mathbb{R}}$, that are *linear* in f, satisfy the involution law $\Phi(f)^* = \Phi(f)$ and the field equation $\Phi(Pf) = 0$, and obey the canonical commutation relations

$$[\Phi(f), \Phi(h)] = i\,\sigma([f], [h])\,\mathbf{1} = i\,E(f, h)\,\mathbf{1}.$$

Concretely $\mathfrak{A}_{\mathrm{CCR}}(V, \sigma) = T_{\mathbb{C}}(V \otimes \mathbb{C}) / \big([\Phi(f), \Phi(h)] - iE(f, h)\mathbf{1}\big)$ is a quotient of the tensor algebra; its elements are the (unbounded) polynomial field observables. The *Weyl C^*-algebra* $\mathcal{W}(V, \sigma)$ is the (bounded) C^*-algebra generated by unitary symbols $W(v)$, $v \in V$, with

$$W(v)^* = W(-v), \qquad W(v)\,W(w) = e^{-\frac{i}{2}\sigma(v,w)}\,W(v + w),$$

formally $W(v) = e^{i\Phi(v)}$; it is the norm completion of a twisted group $*$-algebra of the additive group V. The unbounded generators $\Phi(f)$ live in $\mathfrak{A}_{\mathrm{CCR}}$, not in $\mathcal{W}$; the two are related by the (unbounded) exponential map and share the same GNS/vacuum representation.

Definition 5.3 (Quasi-free vacuum state [EST], [12, 15])**.** A *quasi-free (Gaussian) state* on $\mathfrak{A}_{\mathrm{CCR}}(V, \sigma)$ (equivalently, on $\mathcal{W}(V, \sigma)$) is determined by a real symmetric bilinear form μ on V dominating σ, i.e. $\frac{1}{4}\sigma(v, w)^2 \le \mu(v, v)\,\mu(w, w)$, via $\omega_\mu(W(v)) = e^{-\frac{1}{2}\mu(v,v)}$ on the Weyl generators. Its *two-point function*

$$w_2(f, h) := \omega_\mu\big(\Phi(f)\Phi(h)\big) = \mu([f], [h]) + \tfrac{i}{2}E(f, h)$$

is a positive-type bilinear form with $\operatorname{Im} w_2 = \frac{1}{2}E$. For Minkowski space the vacuum μ is the standard positive-frequency form and w_2 is the Wightman two-point function; the GNS triple $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ recovers the Fock representation with cyclic vacuum Ω_ω.

5.2 The condensed re-encoding

Everything above is functional-analytic data indexed by the test-function space $\mathcal{D}(M)_\mathbb{R}$, which by Proposition 3.2 is a nuclear liquid vector space. We transport each structure map along the fully faithful monoidal embedding $\underline{(-)}$.

Theorem 5.4 (The free-field algebra is a $*$-algebra object in Cond(Vect) [EST]). *Let $V_C := \underline{V} \in \mathrm{Liq}_p(\mathbb{R})$ be the condensed phase space of Definition 5.1 and $\sigma_C\colon V_C \otimes_\mathrm{L} V_C \to \underline{\mathbb{R}}$ the morphism of liquid vector spaces given by the Pauli–Jordan pairing (well defined by Theorem 4.4, as $E \in \mathcal{D}'(M \times M)$). Then:*

1. *the field map $\Phi_C\colon V_C \to \underline{\mathfrak{A}_\mathrm{CCR}(V,\sigma)}$, $f \mapsto \Phi(f)$, is a morphism of liquid vector spaces into the polynomial CCR $*$-algebra of Definition 5.2;*
2. *the multiplication and involution of $\underline{\mathfrak{A}_\mathrm{CCR}(V,\sigma)}$ are morphisms of condensed $\mathbb{C}$-vector spaces, so $\underline{\mathfrak{A}_\mathrm{CCR}(V,\sigma)}$ is an algebra object (a condensed unital $*$-algebra) in* Cond(Vect)*; and*
3. *the CCR relation holds internally: the commutator $[\Phi_C(f), \Phi_C(h)] = i\,\sigma_C(f,h)\,\mathbf{1}$ is an identity of morphisms $V_C \otimes_\mathrm{L} V_C \to \underline{\mathfrak{A}_\mathrm{CCR}(V,\sigma)}$.*

The same holds, at the bounded C^*-level, for the Weyl algebra $\underline{\mathcal{W}(V,\sigma)}$ with its unitary generators $W(v)$. In particular the entire kinematical algebra of the free scalar field is an object of, and its structure maps are morphisms in, the abelian/stable category of condensed vector spaces.

Proof. The tensor-algebra-with-relations construction of $\mathfrak{A}_\mathrm{CCR}(V,\sigma)$ is functorial in the symplectic space (V,σ); applying the fully faithful monoidal functor $\underline{(-)}$ (Propositions 2.4 and 3.2) to the diagram of vector spaces and bilinear maps that defines $\mathfrak{A}_\mathrm{CCR}$ yields a diagram in Cond(Vect). Concretely: $V_C \otimes_\mathrm{L} V_C \to \underline{\mathbb{R}}$ realizes σ by Theorem 4.4 (the Pauli–Jordan kernel is an element of $\mathcal{D}'(M \times M)$, hence a morphism $\mathcal{D}_C(M) \otimes_\mathrm{L} \mathcal{D}_C(M) \to \underline{\mathbb{R}}$ that descends to V_C); the free unital associative $\mathbb{C}$-algebra $T_\mathbb{C}(V_C \otimes \mathbb{C})$ on a condensed vector space is again a condensed algebra, and imposing the (condensed) CCR relation $[\Phi(f), \Phi(h)] - iE(f,h)\mathbf{1}$ – an equality of morphisms – gives a quotient algebra object because Cond(Vect) is abelian (Proposition 2.4) and closed under the needed colimits. The involution is the unique conjugate-linear anti-automorphism fixing the hermitian generators, $\Phi(f)^* = \Phi(f)$, and reversing products; it is a morphism of condensed $\mathbb{C}$-vector spaces. The commutator identity in (3) is the image under $\underline{(-)}$ of the classical CCR, an identity of continuous bilinear maps. Faithfulness of $\underline{(-)}$ ensures no relation is lost or added. The Weyl statement follows identically, replacing the tensor-algebra quotient by the twisted group $*$-algebra of V and its (condensed) C^*-completion, with involution $W(v)^* = W(-v)$. $\square$

Theorem 5.5 (The vacuum is a condensed functional [EST])**.** *The quasi-free vacuum ω_μ of Definition 5.3 is a morphism of condensed $\mathbb{C}$-vector spaces $\omega_\mu\colon \underline{\mathfrak{A}_{\mathrm{CCR}}(V,\sigma)} \to \underline{\mathbb{C}}$ (equivalently on $\underline{\mathcal{W}(V,\sigma)}$), positive and unital as an internal statement; its two-point function $w_2\colon V_C \otimes_{\mathrm{L}} V_C \to \underline{\mathbb{C}}$ is a morphism of liquid vector spaces with $\operatorname{Im} w_2 = \frac{1}{2}\sigma_C$; and the GNS Hilbert space $\mathcal{H}_\omega$, with its cyclic vacuum Ω_ω, has condensed avatar $\underline{\mathcal{H}_\omega} \in \mathrm{Liq}_p(\mathbb{R})$ (any $0 < p \leq 1$), on which $\underline{\mathfrak{A}_{\mathrm{CCR}}(V,\sigma)}$ acts by (unbounded) morphisms and $\underline{\mathcal{W}(V,\sigma)}$ by bounded morphisms of condensed vector spaces.*

Sketch. ω_μ is a linear functional on $\mathfrak{A}_{\mathrm{CCR}}(V,\sigma)$ (continuous, and bounded on the C^*-algebra $\mathcal{W}(V,\sigma)$); its condensed avatar is a morphism $\underline{\mathfrak{A}_{\mathrm{CCR}}} \to \underline{\mathbb{C}}$ by functoriality of $\underline{(-)}$. Positivity ($\omega_\mu(a^*a) \geq 0$) and unitality are equalities/inequalities of real numbers preserved on $(*)$-points, hence hold internally. The two-point function is the composite

$$V_C \otimes_{\mathrm{L}} V_C \xrightarrow{\Phi_C \otimes_{\mathrm{L}} \Phi_C} \underline{\mathfrak{A}_{\mathrm{CCR}}} \otimes_{\mathrm{L}} \underline{\mathfrak{A}_{\mathrm{CCR}}} \xrightarrow{\text{mult}} \underline{\mathfrak{A}_{\mathrm{CCR}}} \xrightarrow{\omega_\mu} \underline{\mathbb{C}},$$

a morphism of liquid vector spaces; its imaginary part is $\frac{1}{2}\sigma_C$ by Definition 5.3. A separable Hilbert space is a Banach space, hence a compactly generated topological vector space whose condensed avatar $\underline{\mathcal{H}_\omega}$ embeds fully faithfully into $\mathrm{Liq}_p(\mathbb{R})$ for every $0 < p \leq 1$ (Banach spaces are liquid); the GNS representation is a $*$-homomorphism of C^*-algebras, hence a morphism of condensed vector spaces on avatars. □

Corollary 5.6 (Free QFT data lives in Cond(Vect) [EST])**.** *Over a fixed globally hyperbolic M, the complete kinematical and quasi-free state data of the free scalar field – phase space, symplectic form, CCR/Weyl algebra, Pauli–Jordan propagator, vacuum functional, two-point function, and GNS representation – form a diagram in the abelian/stable category* Cond(Vect)*, related to the classical picture by the fully faithful embedding $\underline{(-)}$. No information is lost, and the homological algebra of Parts II–III applies verbatim to this diagram.*

5.3 A finite-mode shadow

To make Theorems 5.4 and 5.5 computable we pass to a finite-mode truncation, which is the object realized in the accompanying Haskell code (see the Code availability section). Replacing $\mathcal{D}(M)_{\mathbb{R}}$ by a finite-dimensional symplectic space $(\mathbb{R}^{2n}, \sigma_0)$ with σ_0 the standard symplectic form gives the finite-mode CCR algebra: hermitian generators $q_1, \ldots, q_n, p_1, \ldots, p_n$ with $[q_j, p_k] = i\delta_{jk}\mathbf{1}$, $[q_j, q_k] = [p_j, p_k] = 0$. Finite-dimensional real vector spaces are discrete condensed vector spaces (trivially liquid), so the finite-mode algebra is literally an algebra object in Cond(Vect) and every statement of Theorem 5.4 holds by a finite computation. The Gaussian vacuum two-point function is the $2n \times 2n$ covariance matrix $w_2 = \mu + \frac{i}{2}\sigma_0$; positivity of the state is the matrix inequality $\mu + \frac{i}{2}\sigma_0 \succeq 0$, which the code checks. This finite shadow is a genuine (if truncated) instance of the condensed free field, and it is fully manifold-free – there is no M in a finite mode space – which is why it is the natural computational seed for Section 7.

Example 5.7 (One mode [EST])**.** For $n = 1$ take $V_C = \underline{\mathbb{R}^2}$ with the standard symplectic form and the vacuum covariance $\mu = \frac{1}{2}I_2$:

$$\sigma_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad w_2 = \mu + \tfrac{i}{2}\sigma_0 = \begin{pmatrix} \frac{1}{2} & \frac{i}{2} \\ -\frac{i}{2} & \frac{1}{2} \end{pmatrix}.$$

This w_2 is positive semidefinite with $\det(w_2) = 0$ – the pure-state (minimal-uncertainty) saturation, with Hermitian eigenvalues $\{0, 1\}$. The code computes this eigenvalue data and verifies the CCR $[\hat{q}, \hat{p}] = i$ on a truncated Fock space.

6 Quantum states and operator algebras, condensed-ly

Part II built a condensed sheaf of observable algebras; Part III studied the failure of the state side to glue. Here we record how the operator-algebraic side of quantum theory – nets, states, modular structure – reads in condensed language, at the level of a fixed field theory.

6.1 Condensed nets

Definition 6.1 (Condensed Haag–Kastler net [EST]/[HEU]). Let $\mathcal{R}$ be a poset (or, following Part II, a site) of "regions," where a region may be a bounded open $\mathcal{O} \subset M$ or, more generally, a condensed test object. A *condensed net* is a functor $\mathfrak{A}\colon \mathcal{R} \to \mathrm{Alg}(\mathrm{Cond}(\mathrm{Vect}))$ to condensed unital $*$-algebras satisfying *isotony* ($\mathcal{O}_1 \subseteq \mathcal{O}_2 \Rightarrow \mathfrak{A}(\mathcal{O}_1) \hookrightarrow \mathfrak{A}(\mathcal{O}_2)$) and, when a causal structure is present, *microcausality* (the images of $\mathfrak{A}(\mathcal{O}_1), \mathfrak{A}(\mathcal{O}_2)$ commute for spacelike separated $\mathcal{O}_1, \mathcal{O}_2$). For $\mathcal{R}$ the causally convex regions of a fixed globally hyperbolic M and $\mathfrak{A}$ the free-field net of Theorem 5.4, this is the condensed avatar of the ordinary Haag–Kastler net; that it satisfies isotony and microcausality is [EST] (transport of the classical net), while the assertion that regions *should* be condensed test objects rather than manifold subsets is the [HEU]/[SPEC] content inherited from Part II [2].

Proposition 6.2 (Local condensed algebras [EST]). *For the free scalar field, the local algebra $\mathfrak{A}(\mathcal{O})$ generated by $\{W(v) : \operatorname{supp} v \subseteq \mathcal{O}\}$ is a $*$-subalgebra object of $\underline{\mathcal{W}(V, \sigma)}$ in* Cond(Vect)*; isotony is the functoriality of $\mathcal{O} \mapsto \{v : \operatorname{supp} v \subseteq \mathcal{O}\}$ and microcausality is the vanishing of the Pauli–Jordan pairing σ_C on spacelike-separated supports, an identity of morphisms in* $\mathrm{Liq}_p(\mathbb{R})$.

6.2 States as condensed functionals

Definition 6.3 (Condensed state [EST]). A *state* on a condensed unital $*$-algebra $\mathfrak{A}$ is a morphism of condensed $\mathbb{C}$-vector spaces $\omega\colon \mathfrak{A} \to \underline{\mathbb{C}}$ that is unital and positive as internal statements ($\omega(\mathbf{1}) = 1$, $\omega(a^*a) \geq 0$ on $*$-points). The set of states is a condensed convex set; the vacuum ω_μ of Theorem 5.5 is a distinguished point of it.

This is the condensed avatar of the ordinary state space; Definition 6.3 makes "state" a morphism in the same category as the field, so that the descent analysis of Part III [3] – the failure of a system of local states to glue, measured cohomologically – applies to condensed states without modification. The dictionary is: *fields are kinematic sections, observables their linear duals, states positive functionals on the observable algebra*, all three living in Cond(Vect).

6.3 Tomita–Takesaki, condensed-ly

Remark 6.4 (Modular theory in the condensed language [EST]/[HEU]). Let $(\mathfrak{A}(\mathcal{O}), \Omega)$ be a local von Neumann algebra with cyclic separating vacuum, e.g. a Rindler wedge algebra of the free field. The Tomita operator $S\colon a\Omega \mapsto a^*\Omega$ has polar decomposition $S = J\Delta^{1/2}$ with modular conjugation J and modular operator Δ; the modular flow $\sigma_t(a) = \Delta^{it} a \Delta^{-it}$ is a one-parameter automorphism group, and $J\mathfrak{A}J = \mathfrak{A}'$ (Tomita's theorem). By Bisognano–Wichmann, for a wedge the modular flow is the boost group and J is a PCT reflection. All of this is [EST] operator-algebra over the fixed manifold. Its condensed reading is straightforward and [EST] at the level of avatars: Δ^{it} and J act by morphisms of condensed vector spaces on $\underline{\mathcal{H}_\omega}$, and the modular flow is a one-parameter subgroup $\underline{\mathbb{R}} \to \mathrm{Aut}(\mathfrak{A})$ in Cond(Vect). What is *not* established – and is precisely the Part V question [4] – is whether the geometric content (that σ_t *is* a boost, i.e. the recovery of the wedge geometry from the algebra and state) survives when the manifold is removed; we defer that to Part V and mark it [SPEC] there.

7 Research direction: manifold-free quantum field theory

We now reach the paper's speculative frontier and state it as sharply as the established core permits. The question is whether the free-field success of Section 5 – coefficient data transports losslessly into Cond(Vect) – can be extended to a construction in which *the base carries no presupposed smooth manifold*, and then to interactions. We separate the two.

7.1 An audit of manifold-dependence

The honesty of this Part rests on the following itemized audit (Table 1) of the free-field construction of Section 5. We ask of each ingredient: does it still secretly use the smooth manifold M?

Ingredient	Uses M?	Status / what would remove M
Coefficient category $\mathrm{Liq}_p(\mathbb{R})$, $\otimes_{\mathrm{L}}$, $\underline{\mathrm{Hom}}$	No	[EST]; intrinsic to the analytic ring $(\mathbb{R}, \mathrm{Liq}_p)$.
CCR/Weyl algebra from an abstract symplectic (V, σ)	No	[EST]; needs only a condensed symplectic vector space (Theorem 5.4).
Quasi-free state from an abstract dominating μ	No	[EST]; needs only $\mu \succeq \frac{i}{2}\sigma$ internally (Theorem 5.5).
Finite-mode truncation (Section 5.3)	No	[EST]; already manifold-free (realized in code).
Test-function space $\mathcal{D}(M)$ as *the* symplectic space	Yes	[HEU]; one needs an intrinsic condensed model of $\mathcal{D}(M)$ not built from M's charts.
Klein–Gordon operator $P = \Box_g + m^2$	Yes	[SPEC]; requires an intrinsic (condensed) notion of a differential operator / dynamics without g.
Advanced/retarded Green's operators $E^{\pm}$; causality $J^{\pm}$	Yes	[SPEC]; requires an intrinsic condensed causal structure (Part V/VI).
Pauli–Jordan $E = E^+ - E^-$ as the symplectic form	Partly	[HEU]/[SPEC]; the *form* is abstract (Theorem 5.4), but its *value* is fixed by M's geometry.

Table 1: Manifold-dependence audit of the free-field construction. The algebraic/state-theoretic layer (top four rows) is already manifold-free; the dynamical/causal layer (bottom four rows) still enters through the smooth manifold M.

The audit yields a clean statement: the *algebraic and state-theoretic* layer of free QFT is already manifold-free (top four rows), while the *dynamical/causal* layer – what selects the particular symplectic form and dominating form – still lives on M (bottom four rows). A manifold-free free field is therefore *available* as soon as one specifies an abstract condensed symplectic space (V_C, σ_C) and a dominating μ_C; the open problem is to produce these intrinsically, i.e. to recover the dynamics without first writing $\Box_g$.

7.2 The research hypothesis

[SPEC] Hypothesis: manifold-free quantum field theory

Hypothesis 7.1 (Condensed QFT)**.** There is an analytic ring $\mathcal{A}$ and a class of condensed bases C (condensed anima, not presupposed to be manifolds) such that:

1. a quantum field theory is specified by a condensed symplectic (more generally, factorization) datum over C valued in $\mathrm{D}(\mathcal{A})$ – a condensed symplectic vector space (V_C, σ_C) together with a distinguished dominating form μ_C for the free case, and a suitable deformation for the interacting case;

2. its algebra of observables is an algebra object in $\mathrm{D}(\mathcal{A})$ (a condensed net in the sense of Definition 6.1) and its states are condensed functionals (Definition 6.3);

3. when $C = \underline{M}$ for a globally hyperbolic M and σ_C the Pauli–Jordan form, the theory reduces to the standard free (resp. perturbatively interacting) field theory of Section 5; and

4. the manifold M, where it exists, is recovered as a realization of $(C, \mathcal{A}, V_C, \sigma_C)$ in the sense of Part V [4], rather than presupposed.

Status. Clause (2) is [EST] for the free case (Theorems 5.4 and 5.5); clause (3) is [EST] for the free case and [HEU] for the interacting case; clauses (1) and (4) are [SPEC]. The composite – “a full, ultimately interacting QFT exists manifold-freely and recovers M as a realization” – is [SPEC] under worst-component-wins (Section 1.4), however rigorous the free-field substrate of Section 5 is.

7.3 The interacting frontier

Remark 7.2 (Why interactions are the hard case [HEU]/[SPEC]). The free field is Gaussian: its algebra is generated by a linear field map and its state is fixed by a quadratic form, so *everything* is controlled by the linear/tensor data that Theorems 2.9 and 4.4 handle cleanly. Interactions break this. The Epstein–Glaser/causal perturbation theory and the Costello–Gwilliam factorization-algebra approach both organize the interacting theory as a deformation of the free one controlled by *products* of distributions, whose singularities require renormalization [16, 17]. Two things must be established for Hypothesis 7.1 to reach interactions:

1. a condensed microlocal analysis (Remark 4.5) making the extension of distributional products across the diagonal a controlled operation in the six-functor formalism; and

2. a condensed, manifold-free formulation of the causal/dynamical data that currently enters through $\Box_g$ and $J^{\pm}$ (the bottom rows of Section 7.1).

Neither exists. We therefore present Hypothesis 7.1's interacting clause as a research proposal, and we do not claim that the rigor of the free-field re-encoding transfers to it.

Remark 7.3 (Relation to background independence in AQFT [EST]/[HEU]). Locally covariant QFT (Brunetti–Fredenhagen–Verch) already achieves a strong form of background independence: a QFT is a functor $\mathsf{Loc} \to C^*\text{-Alg}$ over *all* globally hyperbolic spacetimes at once [13]. This is [EST] and is the nearest established precedent for Hypothesis 7.1; the condensed proposal differs by asking to replace the category Loc of manifolds itself by a category of condensed bases, i.e. to be background-*free* rather than background-*independent*. The Gwilliam–Rejzner comparison theorem [17], relating factorization algebras and nets for free theories, is the template a condensed comparison theorem would generalize.

8 Discussion

8.1 What has and has not been shown

The established core of this Part is a single, clean statement: *the linear apparatus of field theory – configuration spaces, test functions, distributions, the free CCR/Weyl algebra, quasi-free states, GNS data – is an object of, and its structure maps are morphisms in, the abelian/stable category of condensed (liquid) vector spaces, faithfully extending the classical picture* (Proposition 3.2, Theorems 4.4, 5.4 and 5.5, and Corollary 5.6). This is genuine mathematical progress in packaging, not merely notation: it places field theory inside a category where homological algebra, descent, and the six-functor formalism are available, which is what Parts II–III and V–VI need. The speculative frontier is equally clean: the base manifold still enters through the dynamical data (Section 7.1), and removing it – Hypothesis 7.1 – is a research program, not a theorem, and is [SPEC] for the interacting theory.

8.2 EST/HEU/SPEC census

Claim	Status	Basis
TVS is not abelian; Cond(Ab) is Grothendieck abelian	[EST]	Propositions 2.3 and 2.4; [5].
Solid/liquid modules are abelian with completed $\otimes_\blacksquare, \otimes_\mathrm{L}$ and analytic-ring six-functor formalism	[EST]	Theorems 2.6 and 2.9 and Definition 2.10; [7, 8, 9].
Nuclear field/test spaces are liquid; Schwartz kernel = tensor–Hom adjunction	[EST]	Proposition 3.2 and Theorem 4.4.
Free scalar CCR/Weyl algebra, vacuum, GNS are objects/morphisms in Cond(Vect)	[EST]	Theorems 5.4 and 5.5 and Corollary 5.6.
Finite-mode condensed CCR (manifold-free)	[EST]	Section 5.3; code.
Solid/liquid distribution theory is the right home for interacting-QFT kernels	[HEU]	Remarks 2.11 and 4.5.
Condensed net/state formalism reduces to Haag–Kastler over a fixed M	[EST]/[HEU]	Definitions 6.1 and 6.3 and Proposition 6.2.
A full (interacting) QFT can be defined manifold-freely, with M recovered as a realization	[SPEC]	Hypothesis 7.1; Remarks 7.2 and 7.3.

Table 2: Epistemic census for Part IV. The mathematical substrate is established; the physical headline claim is speculative, and the composite is capped at [SPEC] by worst-component-wins.

8.3 Relations and non-identifications

Per Section 1.5: the construction is external/analytic (Clausen–Scholze), not synthetic (cohesive HoTT) and not quantum-logical (topos quantum theory). The free-field re-encoding is a theorem; manifold-freeness is a conjecture. The rigor of the linear theory is not claimed for the nonlinear one. We add one further relation: the finite-mode shadow (Section 5.3) is a genuine manifold-free condensed CCR system, so Hypothesis 7.1 is not vacuous – it holds in the finite-dimensional (quantum-mechanical) case, and the open content is precisely the passage to infinitely many modes with a dynamically selected symplectic structure.

8.4 Limitations

Three limitations are worth stating plainly. *First*, "liquid" functional analysis, while abelian and monoidal, is technically heavy; the parameter p and the choice of analytic ring are not canonical, and matching them to physical function spaces (Sobolev, Hadamard) is not fully worked out here. *Second*, the Lorentzian signature enters only through the manifold in Section 5; a genuinely condensed causal structure is deferred to Parts V–VI and is the crux of Section 7.1's bottom rows. *Third*, the interacting theory is untouched beyond the audit of Remark 7.2; the claim of this Part is deliberately confined to the free field plus a clearly labelled conjecture.

9 Conclusion

We asked whether a field $\varphi\colon M \to \mathbb{R}$ can be replaced by a condensed-vector-space–valued object $\varphi\colon C \to \mathrm{Cond}(\mathrm{Vect})$ with the manifold demoted from background to derived datum. The answer has two honest halves. Mathematically, the coefficient side of the substitution is not only possible but advantageous: condensed abelian groups repair the non-abelianness of topological vector spaces (Propositions 2.3 and 2.4); solid and liquid modules supply well-behaved completed tensor products and an internal Hom (Theorems 2.6 and 2.9); nuclear test and field spaces are liquid, with the Schwartz kernel theorem becoming a tensor–Hom adjunction (Proposition 3.2 and Theorem 4.4); and the entire free scalar field – CCR/Weyl algebra, Pauli–Jordan propagator, quasi-free vacuum, GNS data – is a diagram in Cond(Vect) (Theorems 5.4 and 5.5 and Corollary 5.6). Physically, the base side is not yet removable: the dynamical and causal data still enter through the smooth manifold (Section 7.1), and a manifold-free, ultimately interacting quantum field theory (Hypothesis 7.1) is a research program, [SPEC] under the composition discipline of the series. Part IV thus delivers the functional-analytic substrate on which Parts V and VI will attempt to recover geometry as a realization functor and to model gravity as the gluing of informational structures; it delivers, too, a precise ledger (Table 2) of exactly how far the established mathematics reaches and where the speculation begins.

Code availability

The accompanying Haskell package `src/part4-fields-without-background-manifolds/` provides finite, decidable shadows of the EST core, compiling under `-Wall`: a toy condensed/solid vector-space type with a completed tensor product and internal-Hom duality check (`Solid.hs`); a finite-mode CCR/canonical commutation demonstration verifying $[\hat{q}, \hat{p}] = i$ on a truncated Fock space and the Gaussian vacuum covariance $\mu + \frac{i}{2}\sigma \succeq 0$ of Section 5.3 (`CCR.hs`); and a distribution-pairing check realizing the evaluation morphism $\mathcal{D}' \otimes_{\mathrm{L}} \mathcal{D} \to \mathbb{R}$ of Definition 4.2 on a finite basis (`Distribution.hs`), driven by `Main.hs`.

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