A Representation-Theoretic Foundation for Quantum Gravity

Part VI of *Toward a Condensed Representation Theory of Physics*

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Abstract

This is the synthesis paper of a modular seven-part program. It assembles the interfaces built in Parts I–V into a single explicit pipeline,

$$\text{Condensed Objects} \;\to\; \text{Derived Category} \;\to\; \text{Quantum States} \;\to\; \text{Observables} \;\to\; \text{Emergent Geometry},$$

and asks — as a clearly demarcated research program, not a solved problem — whether gravity can be understood as the *gluing of informational structures* rather than as a fundamental metric field. A central methodological correction carried by this revision is that the terminal arrow Observables → Emergent Geometry is *not* atomic: measurement does not produce spacetime directly, but only *structured relational data*, out of which a causal order, a topology, a conformal class, and finally a Lorentzian metric and its dynamics must be *reconstructed* through a chain of intermediate *pregeometric* layers,

$$\text{Meas} \to \text{Rel} \to \text{CausalSite} \to \text{Cond./Derived Pregeometry} \to (X, \preceq, g) \to \text{dynamics}.$$

We label each rung of this ladder [EST]/[HEU]/[SPEC] and anchor it to an established reconstruction theorem — Malament (causal order fixes topology and the conformal metric), causal sets (order + counting → Lorentzian geometry in the continuum limit), Bisognano–Wichmann (modular flow → boost/causal geometry), and Connes (metric from an algebra) — so that the program's central gap becomes four honestly-labeled reconstruction problems, each with a named precedent, rather than a single magic arrow. We are maximally disciplined about the boundary between what is established and what is conjectural, because this is precisely the Part where overclaiming is most tempting. *Established* [EST]: (i) we formalize the collaboration's status calculus as a bounded, totally ordered, idempotent commutative monoid $(\{\mathsf{EST}, \mathsf{HEU}, \mathsf{SPEC}\}, \max, \mathsf{EST})$ and prove the *weakest-link theorem* — the status of any composite pipeline is the join (worst case) of its links, so no composite can be more established than its weakest link; (ii) we exhibit a finite, fully computable instance of the entire pipeline — a finite weighted graph as a discrete condensed object, its Čech cohomology as the derived invariant, a positive weight as a state, and a path metric as the emergent distance — and prove a *finite closure theorem*: this composite is a well-defined functor that satisfies descent (gluing), and its emergent distance coincides with the finite Connes spectral distance of Part V. All of that mathematics is genuinely established and code-backed. *Heuristic* [HEU]: that the assembled pipeline is the correct organizing skeleton for a background-independent quantum theory of gravity, and that microcausality reads as a derived compatibility condition. *Speculative* [SPEC], each boxed and dissected into "what is established / what a proof would require": (A) recovery of classical Lorentzian spacetime as a limiting case; (B) locality as a derived compatibility condition; (C) gravity from the gluing of condensed informational structures. We position each of algebraic QFT, topological QFT, homotopy type theory, derived algebraic geometry, and higher category theory as an *established anchor* the framework specializes or uses, never as evidence that the speculative goals are achieved. A status census closes the paper: the mathematical ingredients are [EST]; the program's headline physical claims are, honestly, [SPEC].

Contents

1 Introduction

1.1 The modular program and its governing perspective

This is Part VI, the *synthesis*, of a modular seven-part program organized by a single perspective:

> *Physics is the study of realizations of condensed mathematical structures.*

The program is deliberately *modular*, not unified. Each Part isolates one interface between condensed mathematics (in the sense of Clausen–Scholze [7, 8]) and physics, states precisely what it establishes, and composes with its predecessors only through explicitly declared hand-offs. The five preceding Parts remove, one at a time, a piece of presupposed geometric background, and then — in Part V — begin to put geometry back as a derived object:

- **Part I** [1] replaced the smooth manifold by the condensed set as the kinematic substrate, exhibiting the fully faithful embedding $\mathrm{Top}_{\mathrm{cg}} \hookrightarrow \mathrm{Cond(Set)}$ of compactly generated spaces into condensed sets (Scholze, Prop. 1.7 [7]), so that manifolds are *faithfully embedded*, not discarded, and the extra room is where Planck-scale structure may live.

- **Part II** [2] attached observables to condensed test objects, obtaining a (co)sheaf of observable algebras on the pro-étale site — a condensed refinement of the Haag–Kastler net and of Costello–Gwilliam factorization algebras — *before* any geometry is posited.
- **Part III** [3] proposed information as the primitive object: entanglement as a compatibility (cocycle) condition in condensed cohomology, with the surface-code $H_1(\Sigma; \mathbb{F}_2)$ theorem and the HaPPY code as its established (non-condensed) precedents.
- **Part IV** [4] demoted the manifold on which a field lives, replacing $\varphi\colon M \to \mathbb{R}$ by a condensed-module-valued object $\varphi\colon C \to \mathrm{Cond}(\mathrm{Vect})$ landing in solid or liquid modules, and audited exactly which ingredients of a free field still secretly invoke M.
- **Part V** [5] recovered geometry as a *realization functor*, organizing Gelfand duality, Tannakian reconstruction, Connes' spectral reconstruction, Tomita–Takesaki/Bisognano–Wichmann, and the cobordism hypothesis into one schema, and isolating the single sharpest gap: every established metric reconstruction is Riemannian, whereas spacetime is Lorentzian.

1.2 What a synthesis paper is, and is not

The task of Part VI is to *recompose* these five interfaces into one object and to ask what, if anything, the composite says about quantum gravity. It is essential to state at the outset what this paper does and does not claim.

The honesty contract of this synthesis

This paper does **not** claim to solve quantum gravity, to derive the Einstein field equations, to construct a Lorentzian emergent spacetime, or to prove that gravity is entanglement. It presents a *research program*: a precise pipeline whose *ingredients* (condensed mathematics, AQFT, TQFT, HoTT, derived algebraic geometry, higher category theory, the Ryu–Takayanagi formula within AdS/CFT) are established mathematics or established-within-their-setting physics, but whose *headline physical interpretation* — that gravity emerges from gluing condensed informational structures — is speculative. We will prove exactly two things and label everything else. Any reader who takes away more than the two boxed [EST] theorems (Theorem 2.3 and Theorem 5.8) has taken away more than we have shown.

The discipline we impose is the collaboration's *status calculus* [38]: every claim is tagged [EST] (established, citable, proved), [HEU] (a physically-motivated heuristic dictionary entry, not itself a theorem), or [SPEC] (a speculative ontological/physical hypothesis of this program), and statuses compose *monotonically, worst-case wins*. Section 2 makes this a small but genuine theorem, which is then the load-bearing device that keeps the rest of the paper honest.

1.3 The assembled pipeline

The object this paper assembles is the following diagram, in which each node is a category (or ∞-category) and each arrow is a construction supplied by one of Parts I–V. We write the per-arrow status beneath each arrow; the composite status is their join.

$$\begin{array}{ccccccccc} \mathrm{Cond}(\mathrm{Set}) & \xrightarrow{\mathbb{Z}[-]} & \mathrm{D}(\mathrm{Cond}(\mathrm{Ab})) & \xrightarrow{\rho} & \text{States} & \xrightarrow{\langle - \rangle} & \text{Obs} & \overset{d}{\dashrightarrow} & (X, \preceq, g) \\ \text{(Part I)} & & \text{(Part IV)} & & \text{(Part III)} & & \text{(Part II)} & & \text{(Part V; (2))} \\ \text{[EST]} & & \text{[EST]} & & \text{[HEU]} & & \text{[HEU]} & & \text{[SPEC] (ladder)} \end{array} \tag{1}$$

Read left to right: a *condensed object* presents the kinematic data (Part I); the free condensed abelian group functor and its derived category supply a *derived/homological invariant* (the solid/liquid derived category of Part IV, the condensed cohomology of Part III); a representation ρ produces *quantum states*; pairing states against the (co)sheaf of *observables* (Part II) yields expectation values; and a realization/distance functor d extracts an *emergent geometry* $(X, \preceq, g)$ (Part V). By the weakest-link theorem the composite is [SPEC], and it is [SPEC] *no matter how rigorous the first two arrows are.* This is the single most important structural fact of the paper and we will not let it be silently upgraded.

The dashed arrow d is drawn dashed on purpose: *it is not atomic.* Measurement does not hand back a spacetime; it hands back structured relational data, and the passage to geometry factors through a chain of intermediate pregeometric layers. We record this de-compression once, here, as the ladder (2), and dissect it rung by rung in Section 3.5:

$$\begin{array}{c} \text{Obs (observable sheaf/cosheaf, Part II)} \\ {\scriptstyle \text{states, effects, instruments}}\Big\downarrow{\scriptstyle \text{[HEU]}} \\ \text{measurement pairings } \langle \rho, E \rangle \\ {\scriptstyle \text{joint measurability}}\Big\downarrow{\scriptstyle \text{[HEU]}} \\ \text{compatibility / descent } [\omega] \in \check{H}^{\geq 1} \text{ (Part III)} \\ {\scriptstyle \text{modular flow / rel. entropy}}\Big\downarrow{\scriptstyle \text{[HEU]}} \\ \text{relational structure} \\ {\scriptstyle \text{Bisognano–Wichmann}}\Big\downarrow{\scriptstyle \text{[SPEC]}} \\ \text{causal / order structure } (S, \preceq) \\ {\scriptstyle \text{Malament}}\Big\downarrow{\scriptstyle \text{[SPEC]}} \\ \text{topology } \tau \\ {\scriptstyle \text{conformal + volume}}\Big\downarrow{\scriptstyle \text{[SPEC]}} \\ \text{metric } (X, \preceq, g) \text{ (Part V)} \\ {\scriptstyle \text{spectral action / Jacobson}}\Big\downarrow{\scriptstyle \text{[SPEC]}} \\ \text{dynamics / Einstein-limit physics} \end{array} \tag{2}$$

The left-hand label of each rung names its mechanism (and, in Section 3.5, its established anchor); the right-hand label is the honest composite status. The upper rungs (down to relational structure) are [HEU] with a large [EST] substrate — and are exactly where the *condensed* contribution is real: assembling continuum-like relational data from finite/profinite probes and extracting its compatibility/cohomology/modular content. The lower rungs (relational $\to$ causal $\to$ topology $\to$ metric $\to$ dynamics) are each [SPEC], but each is now tied to a specific established reconstruction theorem. The blocking work of the whole program is thereby *localized* in this lower segment; it is no longer a single magic arrow "observables $\Rightarrow$ spacetime".

1.4 Contributions

1. **The status calculus as a theorem ([EST]).** We formalize $(\{\mathsf{EST}, \mathsf{HEU}, \mathsf{SPEC}\}, \max)$ as a bounded totally ordered idempotent commutative monoid and prove the weakest-link theorem (Theorem 2.3): composite status is the join of link statuses; composition is associative, commutative, idempotent, monotone, with identity EST and absorbing element SPEC. The accompanying Haskell realizes this as a verified combinator.

2. **A finite closure theorem for the whole pipeline ([EST]).** We instantiate (1) on finite data — a finite weighted graph Γ as a discrete condensed object — and prove (Theorem 5.8) that the composite is a well-defined functor $\mathsf{FinGraph}_w \to \mathsf{Met}$ which (a) satisfies Čech descent for the observable sheaf, (b) has derived invariant $\check{H}^0 = \mathbb{Z}^{\pi_0(\Gamma)}$, $\check{H}^1 = \mathbb{Z}^{b_1(\Gamma)}$, and (c) outputs an emergent distance equal to the finite Connes spectral distance of Part V. This is the finite, rigorous "closure theorem" the program's dossier calls for, and it is code-backed.

3. **A de-compressed terminal ladder ([HEU]/[SPEC], per rung).** We refuse to let the terminal arrow "observables $\to$ emergent geometry" stand as atomic. We replace it ((2), Section 3.5) by an explicit pregeometric ladder Meas $\to$ Rel $\to$ CausalSite $\to$ Pregeometry $\to (X, \preceq, g) \to$ dynamics, label each rung [EST]/[HEU]/[SPEC] honestly, and attach each geometric rung to an established reconstruction theorem — Bisognano–Wichmann (modular flow $\to$ boost/causal geometry), Malament (causal order fixes topology and the conformal metric), causal sets (order + counting $\to$ Lorentzian geometry in the continuum limit), and Connes (metric from an algebra). This localizes the program's blocking work in the relational$\to$causal$\to$topology$\to$metric$\to$dynamics segment and makes "measurements glue, therefore geometry" an *explicitly non*-proof.

4. **Five established anchors, positioned precisely ([EST]).** We locate AQFT, TQFT, HoTT, derived algebraic geometry, and higher category theory each as an established structure the framework *specializes or uses* (Section 4), stating in each case what is borrowed and what is not.

5. **Three speculative goals, dissected ([SPEC]).** For each of (A) classical spacetime as a limit, (B) locality as a derived compatibility condition, (C) gravity from gluing of informational structures, we give a boxed hypothesis, separate the [EST]/[HEU]/[SPEC] layers, and state explicitly *what a proof would require* and what currently blocks it (Section 6).

6. **A weakest-link status census ([SPEC] headline).** A single table (Section 8.2) records the status of every claim and makes the composite [SPEC] status of the program's headline unavoidable to the reader.

1.5 A note on originality and prior art

As recorded in the program's knowledge base, dedicated literature search found *no* existing work combining condensed mathematics (Clausen–Scholze) with quantum gravity, holography, or AQFT beyond the ordinary functional-analytic/operator-algebraic literature that predates condensed mathematics. This absence is a feature to be flagged, not a gap to conceal: Parts IV–VII stake out genuinely unoccupied territory, and *every place the condensed machinery is asked to do physical work beyond what is proved is marked* [SPEC]. We cite only the real, independently verified literature (condensed mathematics, Tannakian duality, AQFT/TQFT, operator algebras, the RT formula) and the three internal prior-work libraries of this collaboration [36, 37, 38], and we take pains to distinguish this program from the related-but-distinct cohesive-HoTT [34] and topos-quantum-theory [35] programs.

1.6 Outline

Section 2 formalizes and proves the status calculus. Section 3 assembles the pipeline stage by stage, recapping each Part's contribution and marking every arrow. Section 4 positions the five established anchors. Section 5 proves the finite closure theorem. Section 6 dissects the three speculative goals. Sections 7–9 give results, discussion (including the status census), limitations, hand-off to Part VII, and conclusion. Section 10 documents the accompanying verified Haskell.

2 The status calculus, made a theorem

Before assembling anything we make precise the device that keeps the assembly honest. The idea is folklore in the collaboration [38] and has a Lean stub for its core monotonicity lemma; here we give a self-contained statement and proof, and (in Section 10) a verified executable realization.

2.1 The warrant order

Definition 2.1 (Warrant set and order)**.** Let $W = \{\mathsf{EST}, \mathsf{HEU}, \mathsf{SPEC}\}$ be the three-element *warrant set*, totally ordered by

$$\mathsf{EST} \; < \; \mathsf{HEU} \; < \; \mathsf{SPEC},$$

read "more established < more speculative". We assign numerical ranks $r(\mathsf{EST}) = 0$, $r(\mathsf{HEU}) = 1$, $r(\mathsf{SPEC}) = 2$, so that $s \le t \iff r(s) \le r(t)$.

The intended reading: EST is the strongest (least speculative, most established) warrant and SPEC the weakest; "the weakest link" of a chain is the one with the *largest* warrant in the order.

Definition 2.2 (Status composition)**.** Define $\oplus \colon W \times W \to W$ by

$$s \oplus t \; = \; \max_{<}(s, t),$$

the join with respect to the total order of Definition 2.1 (equivalently, the element of larger rank). We call $\oplus$ *status composition.*

2.2 The weakest-link theorem

Theorem 2.3 (Weakest-link theorem)**.** *The triple* $(W, \oplus, \mathsf{EST})$ *is a commutative monoid that is moreover* idempotent *and* bounded, *and* $\oplus$ *is* monotone *in each argument. Explicitly, for all* $s, t, u \in W$*:*

(i) (associativity) $(s \oplus t) \oplus u = s \oplus (t \oplus u)$*;*

(ii) (commutativity) $s \oplus t = t \oplus s$*;*

(iii) (identity) $s \oplus \mathsf{EST} = s$*;* $\quad \mathsf{EST}$ *is a two-sided identity;*

(iv) (absorption) $s \oplus \mathsf{SPEC} = \mathsf{SPEC}$*;* $\quad \mathsf{SPEC}$ *is absorbing;*

(v) (idempotence) $s \oplus s = s$*;*

(vi) (monotonicity) $s \leq s' \implies s \oplus t \leq s' \oplus t$*;*

(vii) (worst-case characterization) *for any finite family* $s_1, \dots, s_n \in W$*,*

$$\bigoplus_{i=1}^{n} s_i \;=\; \max_{<}\{s_1, \dots, s_n\},$$

and in particular $\bigoplus_i s_i = \mathsf{EST}$ *iff every* $s_i = \mathsf{EST}$*, while* $\bigoplus_i s_i = \mathsf{SPEC}$ *iff some* $s_i = \mathsf{SPEC}$*.*

Proof. All statements are properties of the binary maximum on a totally ordered set, transported through the rank isomorphism $r\colon (W, <) \xrightarrow{\sim} (\{0, 1, 2\}, <)$. Since r is an order isomorphism, $r(s \oplus t) = \max(r(s), r(t))$. The maximum on a totally ordered set is associative and commutative (both sides of (i), (ii) equal the largest rank among the arguments), has the least element as identity ((iii): $\max(r(s), 0) = r(s)$ since $r(s) \geq 0$) and the greatest element as absorbing element ((iv): $\max(r(s), 2) = 2$ since $r(s) \leq 2$), and is idempotent ((v): $\max(a, a) = a$). Monotonicity (vi) is the standard fact that $a \leq a' \implies \max(a, b) \leq \max(a', b)$. For (vii), induction on n using associativity reduces the iterated $\oplus$ to the n-ary maximum; $\max_i r(s_i) = 0$ iff all $r(s_i) = 0$, and $\max_i r(s_i) = 2$ iff some $r(s_i) = 2$. Reading ranks back through r^{-1} gives the stated warrant identities. ☐

Remark 2.4 (Why a semilattice, not a group). $(W, \oplus)$ has no inverses: once a chain of reasoning contains a [SPEC] link, *no amount of established mathematics downstream can remove it*, because SPEC is absorbing (iv). This is exactly the epistemic content we want. A synthesis paper is under constant temptation to let a long train of rigorous lemmas "pay off" the speculative debt of one physical hypothesis; the monoid structure forbids this arithmetically. Equivalently, $(W, \oplus, \mathsf{EST}, \mathsf{SPEC})$ is a bounded join-semilattice, and status assignment is a monoid homomorphism from the free monoid of pipeline composites to $(W, \oplus)$.

Definition 2.5 (Status of a pipeline)**.** A *pipeline* is a finite composable chain of arrows $A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} A_n$, each arrow f_i carrying a status $\sigma(f_i) \in W$. Its *composite status* is

$$\sigma(f_n \circ \cdots \circ f_1) \;:=\; \bigoplus_{i=1}^{n} \sigma(f_i) \;=\; \max_{<}\{\sigma(f_1), \dots, \sigma(f_n)\}.$$

Corollary 2.6 (The pipeline (1) is [SPEC]). *With the per-arrow labels of* (1)*, and with its terminal arrow* d *de-compressed into the ladder* (2)*, the composite status of the full assembled pipeline* $\mathrm{Cond}(\mathrm{Set}) \to \mathrm{D}(\mathrm{Cond}(\mathrm{Ab})) \to \mathit{States} \to \mathrm{Obs} \to (X, \preceq, g)$ *is*

$$\underbrace{\mathsf{EST} \oplus \mathsf{EST} \oplus \mathsf{HEU} \oplus \mathsf{HEU}}_{\textit{upstream of } \mathrm{Obs}} \oplus \underbrace{\mathsf{HEU} \oplus \mathsf{HEU} \oplus \mathsf{HEU} \oplus \mathsf{SPEC} \oplus \mathsf{SPEC} \oplus \mathsf{SPEC} \oplus \mathsf{SPEC}}_{\textit{ladder } (2)} = \mathsf{SPEC}.$$

Proof. Immediate from Theorem 2.3(vii): the join of the labels is their maximum, SPEC. Unlike in the coarse reading, the maximum is no longer attained at a single "final arrow" but across the entire lower segment of the ladder (2) — the relational→causal→topology→metric→dynamics rungs — each of which is [SPEC] for a separately stated reason (Section 3.5). The absorption law (Theorem 2.3(iv)) makes a single [SPEC] rung sufficient; there are four. □

This corollary is the paper in miniature. Everything after it is either (a) verifying that the first two arrows really are [EST] (Section 3, Section 5), (b) locating the borrowed [EST] anchors (Section 4), or (c) being precise and honest about *why* the last arrow is [SPEC] and what would be required to improve it (Section 6).

3 Assembling the pipeline

We now walk the pipeline (1) node by node, recalling each Part's deliverable and recording precisely what is established at that stage and what is deferred. Throughout, "condensed" means a sheaf on the pro-étale site $*_{\text{proét}}$ of a point — equivalently a sheaf on profinite (or extremally disconnected) sets with finite jointly-surjective covers [7, 9].

3.1 Stage 1 — condensed objects (Part I)

Definition 3.1 (Condensed set). A *condensed set* is a sheaf $T \colon \mathrm{ProFin}^{\mathrm{op}} \to \mathrm{Set}$ on the pro-étale site of a point: $T(\emptyset) = *$, $T(S \sqcup S') = T(S) \times T(S')$, and for a surjection $S' \twoheadrightarrow S$ of profinite sets, $T(S) \to T(S') \rightrightarrows T(S' \times_S S')$ is an equalizer. Condensed abelian groups $\mathrm{Cond}(\mathrm{Ab})$ are the internal abelian group objects; they form a Grothendieck abelian category [7].

Proposition 3.2 (Part I hand-off; Scholze Prop. 1.7). *The functor* $\mathrm{Top} \to \mathrm{Cond}(\mathrm{Set})$, $X \mapsto (S \mapsto \mathrm{Cont}(S, X))$, *is faithful, and fully faithful on compactly generated spaces.* [EST] *In particular any smooth manifold is faithfully embedded as a condensed set with no loss of information; the point of Part I is that* $\mathrm{Cond}(\mathrm{Set})$ *is strictly larger and better behaved (it is closed under limits and colimits and* $\mathrm{Cond}(\mathrm{Ab})$ *is abelian, unlike topological abelian groups).*

This is the [EST] content of the first node. The physical hypothesis of Part I — that Planck-scale spacetime is *better* modeled by a condensed object than a manifold — is [SPEC] and is not used as a premise here; only the embedding is.

3.2 Stage 2 — the derived category (Parts III, IV)

The arrow $\mathbb{Z}[-]\colon \mathrm{Cond}(\mathrm{Set}) \to \mathrm{D}(\mathrm{Cond}(\mathrm{Ab}))$ sends a condensed set T to (the derived image of) the free condensed abelian group $\mathbb{Z}[T]$, landing in the derived ∞-category of condensed abelian groups. This is where the *homological* content lives.

Proposition 3.3 (The derived stage is established)**.** $\mathrm{Cond}(\mathrm{Ab})$ *is a Grothendieck abelian category, so its derived category* $\mathrm{D}(\mathrm{Cond}(\mathrm{Ab}))$ *is a stable* ∞*-category with a natural t-structure; the solid subcategory* $\mathrm{D}(\mathbb{Z}_\blacksquare)$ *and, over* $\mathbb{R}$*, the liquid subcategory are full stable subcategories closed under limits, colimits, and extensions, and carry a symmetric monoidal derived tensor product and internal hom [7, 8].* [EST]

This is precisely the machinery Part IV [4] needed to replace "complete topological vector spaces of fields" (which do not form an abelian category) by solid/liquid modules (which do), and that Part III [3] needed to define condensed cohomology $\check{H}^\bullet$ of a sheaf of observable data. The arrow is [EST] as *mathematics*. What is *not* established is that the resulting condensed cohomology is the right home for a physical entanglement invariant; that is the [SPEC] proposal of Part III, and it enters the pipeline only downstream.

3.3 Stage 3 — quantum states (Part III)

A *state* is produced by a representation ρ of the derived/observable data on a (condensed) Hilbert-type module. In the finite closure model of Section 5 a state is a positive normalized weight; in the intended infinite setting it is a normalized positive linear functional on a condensed operator algebra (a GNS-type datum), with the von Neumann-algebraic and Tomita–Takesaki structure of [19] available. We mark this arrow [HEU]: the *existence* of states on condensed algebras is unproblematic, but the claim that physical quantum states of a gravitating system are faithfully captured by this datum, prior to any geometry, is a heuristic of the program, not a theorem.

3.4 Stage 4 — observables (Part II)

Part II [2] attaches to each condensed test object U an algebra (or module) of observables $\mathrm{Obs}(U)$, and requires *descent*: compatible local observations glue. The correct categorical form is the sheaf/cosheaf fork discussed there — kinematic data restricts (a sheaf; limit-gluing), dynamical/operator data composes (a cosheaf/factorization algebra; colimit-gluing) — and both are condensed refinements of established structures (Haag–Kastler nets [15, 16]; Costello–Gwilliam factorization algebras [17, 18]). The pairing $\langle - \rangle$ of a state against Obs yields expectation values. We mark this arrow [HEU] for the same reason as Stage 3: the descent *mathematics* is [EST] (Theorem 5.8 proves a finite case), but the claim that fundamental observables are condensed-sheaf-valued prior to geometry is a heuristic.

Remark 3.4 (Retaining automorphisms: the gauge principle). A recurring load-bearing fact across the program [36] is that gluing must retain *automorphisms*, not merely coarse values: for a quotient stack $[X/G]$, $\mathrm{Aut}_{[X/G]}(x) \simeq \mathrm{Stab}_G(x)$. In the condensed observable sheaf this says an observable assigned to a condensed "point" (an extremally disconnected profinite set) must remember its gauge automorphisms. The pipeline is therefore properly valued in *groupoids/stacks*, and the descent of Stage 4 is stack descent, not set-level gluing. This does not change the composite status but it is the correct level of generality and we flag it.

3.5 Stage 5 — the terminal ladder: from measurement to geometry

The coarse arrow $d\colon \mathrm{Obs} \to (X, \preceq, g)$ of (1) is the one place the program is most tempted to overclaim, precisely because in the diagram it *looks* atomic: "observables, then geometry." It is not atomic, and treating it as a single arrow is the error this revision corrects. **Measurement does not produce spacetime.** A measurement produces, at most, *structured relational data*: which effects occur with which probabilities in which contexts, which observables commute, how the modular flow of a state acts, and what the entanglement and relative-entropy data are. Spacetime — a set with a causal order, a topology, a conformal class, a Lorentzian metric, and Einstein dynamics — must be *reconstructed* from that relational data through the chain of intermediate *pregeometric* layers displayed in (2), and *each* link is a separate, theorem-shaped obligation. We restate the program's central gap as the arrow chain

$$\underbrace{\mathrm{Meas}}_{\text{pairings}} \xrightarrow{\text{[HEU]}} \underbrace{\mathrm{Rel}}_{\text{relational}} \xrightarrow{\text{[SPEC]}} \underbrace{\mathrm{CausalSite}}_{(S,\preceq)} \xrightarrow{\text{[SPEC]}} \underbrace{\mathrm{Pregeometry}}_{\text{cond./derived}} \xrightarrow{\text{[SPEC]}} (X, \preceq, g) \xrightarrow{\text{[SPEC]}} \text{dynamics}. \tag{3}$$

The condensed contribution is concentrated on the *left*: assembling continuum-like objects from finite/profinite probes, and extracting the compatibility/cohomology/modular data that *feeds* reconstruction. The reconstruction itself — the right of the chain — is licensed, rung by rung, only by the established theorems cited below; where no such theorem exists, the rung is [SPEC], and "measurements glue, therefore geometry" is *explicitly not a proof*. We now walk the ladder. For each rung we name the mechanism, the *established anchor* that makes it theorem-shaped (never a slogan), the honest status, and what a proof of the *background-free, condensed* version would require.

Rung 1 — observables to measurement pairings ([HEU]). The observable (co)sheaf of Stage 4 is paired with states, effects, and instruments in the sense of operational quantum theory and generalized probabilistic theories (GPT): a state ρ and an effect E produce a probability $\langle \rho, E \rangle$, an instrument records post-measurement update. The *pairing itself* is an [EST] bilinear evaluation. What is [HEU] is that these operational primitives are complete and prior to geometry. *What a proof would require:* an operational reconstruction/tomography theorem in the condensed setting, showing the pairings determine the state/effect spaces without presupposed spacetime labels on the outcomes.

Rung 2 — pairings to compatibility / descent data ([HEU]). From the pairings one reads which measurements are *jointly performable* (the quantum marginal problem) and whether local observational data *glue* (descent). The [EST] substrate here is genuine: the finite descent obstruction $[\omega] \in \check{H}^{\geq 1}$ (Proposition 5.3, Lemma 5.5) is a real cohomological invariant, and the marginal problem is a rigorously posed compatibility question. What is

[HEU] is that this compatibility data is the physically correct pregeometric input. *What a proof would require:* a theorem that the condensed descent obstruction computes joint-measurability / marginal-compatibility in the infinite (non-finite-graph) setting.

Rung 3 — compatibility to relational structure ([HEU]). From the compatibility data one extracts the *relational invariants*: correlations; the commutation/noncommutation lattice; context inclusion and Kochen–Specker contextuality; the modular flow σ_t^φ of a faithful normal state φ; and entanglement / relative-entropy data. Each is a *canonically defined* algebraic invariant — the modular automorphism group σ_t^φ is uniquely determined by φ on a von Neumann algebra (Tomita–Takesaki [19]), and Araki's relative entropy is well-defined — so the extraction has a large [EST] substrate. What is [HEU] is that the totality of this relational data is the right pregeometric primitive. *What a proof would require:* a condensed "relational structure" functor with a theorem that it is well-defined and faithful on the relevant class of states. We stress that this relational data manifests its physical richness only in the infinite-dimensional, Type III von Neumann setting, where the modular flow is non-trivial and genuinely thermal/temporal (the very regime of Bisognano–Wichmann and the thermal-time hypothesis); on the finite closures of Section 5 the algebra is Type I and the modular flow is inner and geometrically inert. This is one precise reason the passage to the infinite/continuum limit is physically necessary for the pregeometric data to yield relativistic causal structure, and hence why the finite closure theorem realizes only the algebra→metric leg (Rung 6) and not Rungs 4–5.

Rung 4 — relational to causal / order structure ([SPEC]; [EST] anchor: Bisognano–Wichmann). This is the first genuinely pregeometric-to-geometric rung, and it has a sharp established anchor. The Bisognano–Wichmann theorem [24] states that, for a Wightman/AQFT vacuum, the Tomita–Takesaki modular flow of the algebra of a Rindler wedge *is* the geometric one-parameter group of Lorentz boosts preserving that wedge, and the modular conjugation implements the PCT/wedge reflection. Thus *modular flow determines a piece of causal geometry* — a boost, hence a wedge, hence a causal order — and this is a theorem, [EST] in its setting; the Connes–Rovelli thermal-time hypothesis [25] extends it into a [HEU] program for reading a canonical time off a state. What is [SPEC] is recovering a *full* causal order $(S, \preceq)$ from relational/modular data *without* presupposing the vacuum and its background wedge structure. *What a proof would require:* a modular-covariance / Bisognano–Wichmann analogue in the condensed, background-free setting, deriving a causal order from modular data alone.

Rung 5 — causal order to topology ([SPEC]; [EST] anchor: Malament). Given a causal order, topology is nearly forced. *Malament's theorem* [22]: for a distinguishing spacetime (e.g. globally hyperbolic), the chronological/causal order determines the manifold topology, the smooth structure, and the *conformal* class of the metric — everything but a conformal (volume) factor. So "order ⇒ topology" is a theorem, [EST] in the direction of determination. In the discrete/reconstructive direction, causal-set theory [23] takes a locally finite partial order and reconstructs a Lorentzian geometry in the continuum limit — [EST] in controlled cases (faithful sprinklings) and a genuine discrete-to-continuum precedent. What is [SPEC] is that the causal order *our* pipeline extracts (from condensed relational data) is of the type these theorems require, and that its continuum limit exists. *What a proof would require:* a theorem that the extracted $(S, \preceq)$ is (asymptotically) a distinguishing causal order / faithfully embeddable causal set, so that Malament or causal-set reconstruction applies.

Rung 6 — topology to metric ([SPEC]; [EST] anchors: Malament conformal class, causal-set volume, Connes spectral distance). By Malament the metric splits into a *conformal class* (fixed by the causal order, [EST]) and a *volume/scale* factor. The scale is supplied, in causal-set theory, by *counting*: spacetime volume = number of elements (order + number $\to$ geometry), [EST] in controlled cases [23]. Independently, the Riemannian "algebra $\to$ metric" leg is realized by Connes' spectral distance $d_D(p, q) = \sup\{|f(p) - f(q)| : \|[D, f]\| \leq 1\}$ (Theorem 5.7; Part V), which is [EST] but Riemannian. Assembling a conformal class, a volume, and a spectral distance into a single *Lorentzian* metric is Part V's frontier L (Riemannian $\neq$ Lorentzian) and is [SPEC]. *What a proof would require:* a Lorentzian spectral-triple / conformal-plus-volume reconstruction theorem, of which none currently exists.

Rung 7 — metric to dynamics / Einstein limit ([SPEC]; [EST] templates: spectral action, Jacobson, Regge). Finally, dynamics: that the reconstructed geometry obeys (a limit of) the Einstein equations. The nearest established templates are the Chamseddine–Connes spectral action [21], from which an Einstein–Hilbert term falls out asymptotically; Jacobson's thermodynamic derivation of the Einstein equation as an equation of state from the Clausius relation and horizon entropy [26]; and the Regge / large-spin limit. These are templates for *how* Einstein dynamics can emerge from an algebraic / spectral / thermodynamic principle, not instances of *this* pipeline's claim, which is [SPEC]. *What a proof would require:* a derivation that extremizing or varying the condensed gluing data reproduces an Einstein-equation limit.

Remark 3.5 (Where the blocking work now lives). The de-compression localizes the difficulty. Rungs 1–3 (measurement $\to$ relational) are [HEU] with a large [EST] substrate, and are exactly where the *condensed* contribution is real: assembling continuum-like relational data from finite/profinite probes and extracting its compatibility/cohomology/modular content. Rungs 4–7 (relational $\to$ causal $\to$ topology $\to$ metric $\to$ dynamics) are each [SPEC], but each is now attached to a *specific* established reconstruction theorem — Bisognano–Wichmann, Malament, causal sets, Connes, the spectral action — that says precisely what a background-free, condensed analogue would have to prove. The gap is no longer a single magic arrow "observables $\Rightarrow$ spacetime"; it is four honestly-labeled reconstruction problems, each with a named [EST] precedent and a stated obligation. In particular the two independent Part V frontiers survive intact and are now *placed*: the absence of a condensed Tannakian reconstruction (frontier C) obstructs Rungs 3–4, and Riemannian $\neq$ Lorentzian (frontier L) obstructs Rung 6. By Corollary 2.6 the composite remains [SPEC].

$$
\begin{tikzcd}
\mathrm{Cond(Set)} \arrow[r, "\mathbb{Z}[-]"] \arrow[d, dashed, "\mathrm{I}"'] & \mathrm{D(Cond(Ab))} \arrow[r, "\rho"] \arrow[d, dashed, "\mathrm{III/IV}"'] & \mathrm{States} \arrow[r, "\langle - \rangle"] & \mathrm{Obs} \arrow[r, dashed, "d"] \arrow[d, dashed, "\mathrm{II}"] & (X, \preceq, g) \arrow[d, dashed, "\mathrm{V}"] \\
\begin{array}{c}\text{cond. kin.}\\ \scriptstyle \hookleftarrow \text{ manifolds}\end{array} & \begin{array}{c}\check{H}^{\bullet}\\ \scriptstyle \text{solid/liquid}\end{array} & & \begin{array}{c}\text{descent}\\ \scriptstyle \text{Part II}\end{array} & \begin{array}{c}\text{recon.}\\ \scriptstyle \text{ladder (2)}\end{array}
\end{tikzcd}
$$

Figure 1: The assembled pipeline of (1) with the responsible Part beneath each interface. The horizontal arrows are the pipeline; the dashed vertical annotations name the Part that establishes each stage. Arrows 1–2 are [EST]; arrows 3–4 are [HEU]; the terminal arrow d (dashed) is *not atomic* — it is the reconstruction ladder (2), whose lower rungs (relational→causal→topology→metric→dynamics) are each [SPEC]. The composite is [SPEC] (Corollary 2.6).

4 The framework as a specialization of established anchors

A synthesis is only as credible as its relationship to known mathematics. We position five established areas, each as an *anchor* the framework *specializes or uses* — never as evidence for the speculative goals. In each case we state the anchor precisely, say what the pipeline borrows, and name what the pipeline does *not* get for free.

4.1 Algebraic quantum field theory

Definition 4.1 (Haag–Kastler net; locally covariant QFT)**.** A *Haag–Kastler net* assigns to each bounded region $\mathcal{O} \subset \mathbb{R}^{1,3}$ a unital C^*-algebra $\mathcal{A}(\mathcal{O})$ with isotony ($\mathcal{O}_1 \subseteq \mathcal{O}_2 \Rightarrow \mathcal{A}(\mathcal{O}_1) \subseteq \mathcal{A}(\mathcal{O}_2)$) and Einstein causality ($[\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)] = 0$ for spacelike-separated regions). A *locally covariant QFT* [16] is a functor $\mathcal{A} \colon \mathsf{Loc} \to C^*\text{-Alg}$ from globally hyperbolic spacetimes and causal isometric embeddings to C^*-algebras satisfying isotony and causality functorially. [EST]

What the pipeline uses. Stage 4 (observables, Part II) is a condensed refinement of the net: $\mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$ becomes $U \mapsto \mathrm{Obs}(U)$ for U a condensed test object, and isotony/causality become sheaf/cosheaf descent and a microcausality compatibility condition. Local covariance — background independence at the level of *which* spacetime one works over — is the established precedent for building observables before fixing geometry. **What it does not get.** AQFT presumes a globally hyperbolic Lorentzian background *per object*; the condensed refinement aspires to drop even that, which no theorem yet licenses. So AQFT anchors Stage 4 as [EST] and bounds exactly how much of the geometry-free ambition is already underwritten (none of the Lorentzian part).

4.2 Topological quantum field theory and the cobordism hypothesis

Theorem 4.2 (Cobordism hypothesis; Baez–Dolan, Lurie [11, 12])**.** *Fully extended TQFTs $Z \colon \mathsf{Bord}_n \to \mathcal{C}$ valued in a symmetric monoidal (∞, n)-category with duals are classified by* *fully dualizable objects, via $Z \mapsto Z(\mathrm{pt})$.* [EST] *(cited, not reproduced)*

What the pipeline uses. The cobordism hypothesis is the first fully rigorous instance of "physics as a functor": a field theory *is* a monoidal functor, rigidly reconstructible from minimal data. Our pipeline (1) is, structurally, a chain of functors, and Part V's realization functor is modeled on "Z determined by $Z(\mathrm{pt})$". **What it does not get.** TQFT is *topological* — no metric, no local dynamics. It licenses the *functorial form* of the pipeline but says nothing about recovering a metric, let alone the Einstein dynamics; using it as an anchor is a statement about *shape*, not content.

4.3 Homotopy type theory and univalence

Principle 4.3 (Univalence as the Equivalence axiom [33])**.** Voevodsky's univalence axiom $(A = B) \simeq (A \simeq B)$ makes equivalent structures internally indistinguishable, discharging the program's *Equivalence axiom*: a realization functor cannot depend on more than the equivalence class of its input. The distinction between the groupoid quotient $X/\!\!/G$ (a 1-type, retaining stabilizers) and the set quotient X/G (a 0-type) is the type-theoretic form of Remark 3.4. [EST] as mathematics.

What the pipeline uses. HoTT supplies the internal justification that our constructions are invariant under equivalence and that gauge/automorphism data must be retained (the $X/\!\!/G$ vs. X/G distinction). **What it does not get, and a non-identification.** Condensed mathematics is an *external, analytic* theory (sheaves on profinite sets); cohesive HoTT [34] is an *internal, synthetic* theory (modal type theory). They aim near the same target but are technically distinct, and we do not conflate them (Section 8.3).

4.4 Derived algebraic geometry

Proposition 4.4 (Derived and spectral geometry [11, 13, 14])**.** *Stable ∞-categories and their t-structures (Lurie), derived/spectral schemes locally modeled on E_∞-rings, and PTVV shifted symplectic structures (the derived critical locus of a function carries a canonical (-1)-shifted symplectic structure, recovering the BV antibracket) are established [14].* [EST]

What the pipeline uses. Stage 2 lives in a stable ∞-category; the passage to *condensed derived* algebraic geometry (animated condensed rings, $\mathrm{D}(\mathrm{Cond}(\mathrm{Ab}))$) is the natural home for combining Parts III–V. Derived critical loci are the established template for the classical field theory Part IV re-expresses. **What it does not get.** A worked-out "condensed derived algebraic geometry for Lorentzian physics" does not exist in the literature; DAG anchors the *homological* stage and no more.

4.5 Higher category theory and ∞-topoi

What the pipeline uses. The condensed ∞-topos of condensed anima is the ambient $(\infty, 1)$-category in which every stage lives, and ∞-topos descent [11] is the correct form of the gluing that Stage 4 requires. The recent constructions of the condensed homotopy type of a scheme and shape theory for condensed anima [10] indicate the machinery is being actively

Anchor	What the pipeline borrows ([EST])	What it does ***not*** get
AQFT	observables-before-geometry; isotony/causality; local covariance	Lorentzian background is still presumed per object
TQFT	the functorial form "physics is a functor"; rigidity from a point	no metric, no local dynamics
HoTT	equivalence-invariance; retain automorphisms ($X/\!\!/G$)	internal $\neq$ external; not cohesive HoTT
DAG	stable ∞-cat. homological stage; (-1)-shifted symplectic	no condensed-derived Lorentzian geometry exists
∞-topoi	the ambient category; descent/gluing	physical application is the [SPEC] frontier

Table 1: The five established anchors, each a structure the framework *specializes or uses*. Every entry in the middle column is [EST]; the pipeline's speculative content is precisely what is *not* underwritten by the right column.

developed. **What it does not get.** These are arithmetic/geometric developments; their application to *physical* (Lorentzian, dynamical) settings is exactly the program's [SPEC] frontier.

5 A finite closure theorem for the pipeline

We now discharge the promise that the first arrows of (1) are genuinely [EST] by instantiating the *entire* pipeline on finite data and proving a closure theorem: the composite is a well-defined, descent-satisfying functor whose emergent distance is the finite Connes spectral distance of Part V. This is the "closure theorem" the program's dossier calls for, at the finite level where every step is a genuine theorem. The physical interpretation of the output remains [SPEC]; the *mathematics* of the composite is [EST].

5.1 The finite condensed object

Definition 5.1 (Discrete condensed set of a finite set)**.** Every finite set V is a condensed set via the discrete embedding $\underline{V} = (S \mapsto \mathrm{Cont}(S, V))$ of Proposition 3.2; on profinite S this is locally constant V-valued functions. Finite sets thus sit inside Cond(Set) as the *discrete* objects, and any finite diagram of finite sets is a finite diagram of condensed sets. [EST]

Definition 5.2 (Finite weighted graph as pipeline input)**.** Let $\Gamma = (V, E, w)$ be a finite connected-or-not graph: a finite vertex set V, a finite edge set $E \subseteq \binom{V}{2}$, and a weight

$w \colon E \to \mathbb{R}_{>0}$. We regard the vertex set as the discrete condensed object $\underline{V}$, and the edge set as a distinguished finite *cover* datum: the nerve $\mathrm{N}(\Gamma)$ has 0-simplices V and 1-simplices E. Write $\mathsf{FinGraph}_w$ for the category of finite weighted graphs and weight-nonincreasing simplicial maps.

5.2 The derived invariant

The Stage-2 arrow $\mathbb{Z}[-]$ applied to the nerve produces its simplicial chain complex; dually, the constant-sheaf Čech complex of the cover computes cohomology. For a graph (a 1-dimensional simplicial complex) this is elementary and complete. Strictly, the explicit two-term complex $0 \to A^V \to A^E \to 0$ written below is the *simplicial cochain complex* of $\mathrm{N}(\Gamma)$; it computes Čech cohomology precisely because it coincides with the Čech complex of the open cover of $|\mathrm{N}(\Gamma)|$ by the open stars of the vertices, whose nerve is $\mathrm{N}(\Gamma)$ itself. We use "Čech" and "simplicial" interchangeably in this finite setting, with this identification understood.

Proposition 5.3 (Čech cohomology of the graph nerve)**.** *Let A be an abelian group and $\underline{A}$ the constant sheaf on $\mathrm{N}(\Gamma)$. The Čech complex is $0 \to A^V \xrightarrow{\delta} A^E \to 0$ with $(\delta f)(\{u,v\}) = f(v) - f(u)$ (fixing an orientation on each edge). Then*

$$\check{H}^0(\Gamma; A) \;\cong\; A^{\pi_0(\Gamma)}, \qquad \check{H}^1(\Gamma; A) \;\cong\; A^{b_1(\Gamma)}, \qquad b_1(\Gamma) = |E| - |V| + |\pi_0(\Gamma)|,$$

where $\pi_0(\Gamma)$ is the set of connected components and b_1 the first Betti (cyclomatic) number. [EST]

Proof. $\ker\delta$ consists of functions constant on each connected component, so $\check{H}^0 = \ker\delta \cong A^{\pi_0}$. The complex has Euler characteristic $\dim_A A^V - \dim_A A^E = |V| - |E|$ (for A a field, or ranks for $A = \mathbb{Z}$ on the free parts), and $\dim \check{H}^0 - \dim \check{H}^1 = |V| - |E|$, whence $\dim \check{H}^1 = |E| - |V| + |\pi_0| = b_1$. The identification $\check{H}^1 \cong A^{b_1}$ is the standard computation of the first cohomology of a graph, i.e. of the cycle space of Γ. The coefficient group A enters only trivially: a graph is a 1-dimensional CW complex with free, torsion-free integral (co)homology $\check{H}^i(\Gamma; \mathbb{Z}) \cong \mathbb{Z}^{b_i}$, so the universal coefficient theorem gives $\check{H}^i(\Gamma; A) \cong \check{H}^i(\Gamma; \mathbb{Z}) \otimes_{\mathbb{Z}} A \cong A^{b_i}$ for *every* abelian group A, with no restriction to fields; the Euler-characteristic count above is simply the quickest route to b_i. □

Thus the derived invariant of the finite pipeline reads off the topology of Γ: $\check{H}^0$ counts components (the "superselection sectors" of the toy) and $\check{H}^1$ counts independent loops (the "obstruction to global gluing of locally constant observables"). This is the finite, rigorous shadow of Part III's "information as cohomology".

5.3 Descent for the observable sheaf

Definition 5.4 (Finite observable sheaf)**.** For the finite site of V (subsets, with covers = finite jointly-surjective families), let $\mathrm{Obs}(U) = \mathrm{Map}(U, A)$ be the A-valued functions on $U \subseteq V$, with restriction the usual restriction of functions. This is the finite shadow of Part II's condensed observable sheaf.

Lemma 5.5 (Finite descent)**.** Obs $=$ Map$(-, A)$ *is a sheaf on the finite site of* V*: for every cover* $\{U_i\}$ *of* U*, the sequence*

$$\mathrm{Obs}(U) \;\longrightarrow\; \prod_i \mathrm{Obs}(U_i) \;\rightrightarrows\; \prod_{i,j} \mathrm{Obs}(U_i \cap U_j)$$

is an equalizer. [EST]

Proof. A family $(f_i)_i$ with $f_i \in \mathrm{Map}(U_i, A)$ agreeing on overlaps, $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$, defines a well-defined function $f\colon U \to A$ by $f(x) = f_i(x)$ for any i with $x \in U_i$ (such i exists since $\{U_i\}$ covers U; the value is independent of the choice by the overlap condition). Then $f|_{U_i} = f_i$, and f is the unique such function since a function is determined by its values. Hence the map $\mathrm{Obs}(U) \to \{\text{compatible families}\}$ is a bijection, i.e. the sequence is an equalizer. □

Lemma 5.5 is deliberately elementary; its role is to make Part II's descent claim a *proved* statement at the finite level, so that Stage 4's [EST] mathematical substrate is not merely asserted. The passage to genuinely condensed test objects (extremally disconnected profinite sets, where local constancy is nontrivial) is where the real work of Part II lies; the finite case is its skeleton.

5.4 State, observable, and emergent distance

From here on we specialize the coefficient group to $A = \mathbb{R}$ (or $A = \mathbb{C}$), so that the observable algebra $\mathcal{A} = A^V$ is a commutative C^*-algebra, $\mathcal{H} = A^V \oplus A^E$ is a genuine (finite-dimensional) Hilbert space, and the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ and the norms and suprema of Theorem 5.7 are strictly well-defined. The cohomology computation of Proposition 5.3 is unaffected by this specialization (it holds for every A).

Definition 5.6 (State and emergent distance)**.** A *state* on Γ is the datum of the positive edge weights w, read as a length metric on the nerve. The Stage-5 arrow d assigns to Γ the *path metric*

$$d_\Gamma(u, v) \;=\; \inf\Big\{ \textstyle\sum_{e \in P} w(e) \;:\; P \text{ a path from } u \text{ to } v \Big\},$$

with $d_\Gamma(u, v) = +\infty$ if u, v lie in different components. This is the emergent geometry (V, d_Γ).

Theorem 5.7 (Path metric $=$ finite Connes spectral distance)**.** *Let* $(\mathcal{A}, \mathcal{H}, D)$ *be the finite spectral triple of the weighted graph* Γ *of Part V, with* $\mathcal{A} = A^V$ *acting diagonally on* $\mathcal{H} = A^V \oplus A^E$ *and* D off-diagonal *with respect to this grading:* D *maps* $A^V \to A^E$ *(and* $A^E \to A^V$ *by self-adjointness) via the weighted incidence operator with entries* $1/w(e)$*. Consequently, for* $f \in \mathcal{A} = A^V$ *the commutator* $[D, f]$ *is again off-diagonal with edge entries* $(f(u) - f(v))/w(e)$*, which is what isolates the weighted edge differences below. Then for all* $u, v \in V$*,*

$$d_D(u, v) \;=\; \sup\big\{ |f(u) - f(v)| : f \in \mathcal{A},\; \|[D, f]\| \le 1 \big\} \;=\; d_\Gamma(u, v).$$

[EST] (Part V, via finite Kantorovich–Rubinstein duality.)

Proof sketch. The commutator norm $\|[D, f]\|$ equals $\max_{e=\{u,v\}} |f(u) - f(v)|/w(e)$, so the constraint $\|[D, f]\| \leq 1$ is exactly $|f(u) - f(v)| \leq w(e)$ for every edge, i.e. f is 1-Lipschitz for the path metric. The supremum of $|f(u) - f(v)|$ over 1-Lipschitz f is $d_\Gamma(u, v)$ by the Kantorovich–Rubinstein duality (the discrete Monge–Kantorovich theorem): the extremal f is $x \mapsto d_\Gamma(u, x)$. This is the finite instance proved in full in Part V [5], whose code we reuse. □

5.5 The closure theorem

Theorem 5.8 (Finite pipeline closure)**.** *The assignment*

$$\Gamma \longmapsto \Big(\underbrace{\underline{V}}_{\textit{Stage 1}} \rightsquigarrow \underbrace{\check{H}^\bullet(\Gamma; A)}_{\textit{Stage 2}} \rightsquigarrow \underbrace{w}_{\textit{Stage 3–4}} \rightsquigarrow \underbrace{(V, d_\Gamma)}_{\textit{Stage 5}} \Big)$$

is a well-defined functor $\mathcal{P}\colon \mathsf{FinGraph}_w \to \mathsf{Met}$ *to (extended) metric spaces, and it satisfies the following, all* [EST]*:*

(a) (descent) *the Stage-4 observable presheaf* $\mathrm{Obs} = \mathrm{Map}(-, A)$ *is a sheaf on the finite site of* V *(Lemma 5.5);*

(b) (derived invariant) *the Stage-2 invariant is* $\check{H}^0 = A^{\pi_0(\Gamma)}$, $\check{H}^1 = A^{b_1(\Gamma)}$ *(Proposition 5.3); in particular the number of superselection sectors and independent gluing obstructions of the toy are computed by the derived stage;*

(c) (emergent metric) *the Stage-5 output equals the finite Connes spectral distance,* $d_\Gamma = d_D$ *(Theorem 5.7);*

(d) (functoriality/monotonicity) *a weight-nonincreasing simplicial map* $\Gamma \to \Gamma'$ *induces a distance-nonincreasing map* $(V, d_\Gamma) \to (V', d_{\Gamma'})$ *and a map on* $\check{H}^\bullet$*.*

Consequently the finite pipeline is a genuine composite of established constructions, and its composite mathematical *status is* [EST]*. Its composite* physical *status — the reading of* (V, d_Γ) *as an emergent spacetime — is* [SPEC]*, by Corollary 2.6.*

Proof. Well-definedness and functoriality: Stage 1 is the fully faithful discrete embedding (Definition 5.1); Stage 2 is the constant-sheaf Čech functor, natural in simplicial maps; Stages 3–4 record the positive weight and pair it against Obs; Stage 5 is the path-metric functor, which is functorial on weight-nonincreasing maps because such a map cannot lengthen a shortest path (each edge image has weight $\leq$ its preimage, so $d_{\Gamma'}(F(u), F(v)) \leq d_\Gamma(u, v)$). Claims (a), (b), (c) are Lemma 5.5, Proposition 5.3, Theorem 5.7 respectively. Claim (d) is the functoriality just established for the metric together with functoriality of Čech cohomology. Each ingredient is a theorem, so by Theorem 2.3(vii) the composite mathematical status is $\mathsf{EST} \oplus \cdots \oplus \mathsf{EST} = \mathsf{EST}$. The physical reading of the output attaches the [SPEC] label of the Stage-5 *physical* arrow (Section 3.5), and $\mathsf{EST} \oplus \mathsf{SPEC} = \mathsf{SPEC}$. □

Remark 5.9 (What the closure theorem does and does not vindicate)*.* Theorem 5.8 shows the pipeline is not vaporware: on finite data it is a completely rigorous chain of functors, computed by the accompanying code. It vindicates the *form* of (1) and the [EST] status of its first two arrows. Located on the terminal ladder (2), it realizes *only* the algebra→metric leg (Rung 6, the Riemannian Connes distance) on an input whose causal order is trivial: it does not touch Rungs 4–5 (relational → causal → topology, where the Bisognano–Wichmann and Malament analogues would be needed) or Rung 7 (dynamics). It does *not* vindicate any physical claim: the input is a finite graph, not a quantum-gravitational state; the output is a finite *Riemannian* metric space, not a Lorentzian spacetime; the derived invariant is graph topology, not physical entanglement. The theorem is the honest maximal EST payload of a synthesis whose headline remains [SPEC].

6 The three speculative goals, dissected

We now confront the program's three headline aspirations. For each we give a boxed [SPEC] hypothesis, separate the [EST]/[HEU]/[SPEC] layers, and state *what a proof would require* and what currently blocks it. This section contains no theorems about physics; its purpose is to make the speculation precise and its cost explicit.

6.1 Goal A — classical spacetime as a limiting case

Goal A [SPEC]

There is a filtered/completion procedure lim on condensed-representation data such that, in an appropriate regime, the emergent geometry $(X, \preceq, g) = d(\mathrm{Obs})$ of (1) — assembled through the reconstruction ladder (2) — converges to a smooth Lorentzian manifold satisfying (to leading order) the Einstein field equations.

[EST] **layer.** Semiclassical/limit constructions *do* recover geometry in specific established settings: the large-spin (Regge) limit of spin-network/spin-foam amplitudes; Connes' spectral reconstruction recovering a Riemannian spin manifold from a spectral triple [20]; the thermodynamic/GNS limit of operator algebras. These are theorems *in their own settings*.

[HEU] **layer.** That the condensed pipeline admits an analogous limit — a filtered colimit of finite condensed objects (like Theorem 5.8's graphs) whose emergent path metrics converge, in a Gromov–Hausdorff-type sense, to a smooth metric — is a plausible heuristic supported by the finite closure theorem (finite metric spaces *do* converge to manifolds in known examples).

[SPEC] **layer.** That such a limit yields a *Lorentzian* manifold obeying Einstein's equations is speculative and inherits the Riemannian $\neq$ Lorentzian gap.

How it factors through the ladder (2). Goal A is exactly the demand that Rungs 5–7 (causal → topology → metric → dynamics) admit a *continuum limit*: a cofiltered system of finite causal orders whose Malament topologies and causal-set volumes converge to a smooth Lorentzian metric obeying an Einstein limit. It is not a single-step claim, and its cost is distributed across those rungs.

What a proof would require. (i) A precise limiting system: a cofiltered diagram of condensed objects and a topology in which emergent distances converge; (ii) a convergence theorem $d_{\Gamma_n} \to g$ to a smooth (pseudo-)metric (Rung 6); (iii) a Lorentzian upgrade recovering signature and causal structure (Rungs 4–6, Part V frontier L), for which not even a Riemannian condensed reconstruction theorem (frontier C) yet exists; (iv) an Einstein-limit derivation (Rung 7). Composite status of Goal A: [SPEC].

6.2 Goal B — locality as a derived compatibility condition

Goal B [SPEC]

Microcausality/locality is equivalent to the vanishing of a compatibility (cocycle) class in condensed cohomology: local observable data glue to a global observable iff a derived obstruction $[\omega] \in \check{H}^{\geq 1}$ vanishes, and this cohomological condition reproduces Einstein causality in the appropriate limit.

[EST] **layer.** Einstein causality $[\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)] = 0$ for spacelike separation is an axiom of AQFT [15, 16], and functorial local covariance [16] is established. The vanishing of a Čech obstruction class as the exact condition for local-to-global gluing is a theorem (Proposition 5.3, Lemma 5.5): in the finite toy, locally constant observables glue globally iff the relevant class in $\check{H}^1(\Gamma; A)$ vanishes.

[HEU] **layer.** Reading microcausality itself — a *Lorentzian, causal* statement — as such a derived compatibility condition is a heuristic dictionary entry: it is structurally natural (both are "compatibility of local data") but not a theorem.

[SPEC] **layer.** That *all* of relativistic locality, including causal order, is captured by a condensed-cohomological vanishing condition *without a background metric* is speculative.

How it factors through the ladder (2). Goal B lives at the seam between Rungs 2–3 (compatibility/descent → relational, its [EST]/[HEU] substrate) and Rung 4 (relational → causal/order, its [SPEC] content). The vanishing of $[\omega] \in \check{H}^{\geq 1}$ is a genuine compatibility statement at Rungs 2–3; upgrading it to *microcausality*, a causal statement, is precisely the Rung 4 step (relational → causal), where Bisognano–Wichmann is the only anchor and supplies it only in the vacuum. The goal must not be read as skipping from Rung 2 straight to a causal conclusion.

What a proof would require. (i) A condensed-cohomological definition of a "causal compatibility class" attached to a pair of condensed regions; (ii) a theorem that its vanishing is equivalent to spacelike commutativity in the AQFT limit; (iii) a background-free formulation in which causal order is itself derived, not presumed — i.e. a discharge of Rung 4. Step (iii) is blocked by the absence of any condensed Lorentzian/causal structure. Composite status of Goal B: [SPEC] (with an unusually large [EST] substrate — the gluing mathematics of Rungs 2–3 — which must not be mistaken for the causal claim of Rung 4).

6.3 Goal C — gravity from the gluing of informational structures

Goal C [SPEC] — the program's headline

Gravity is not a fundamental metric field but the large-scale manifestation of how condensed informational structures glue: the emergent metric g is determined by entanglement/compatibility data (a condensed-cohomological refinement of the Ryu–Takayanagi relation and of ER = EPR), and Einstein dynamics is the statement that this gluing is consistent.

[EST] **layer.** Within AdS/CFT and its toy models the mathematics is genuine: the Ryu–Takayanagi formula $S(A) = \mathrm{Area}(\gamma_A)/4G_N$ holds in its stated setting, as does its quantum (Faulkner–Lewkowycz–Maldacena) bulk-entropy correction $+\,S_{\mathrm{bulk}}$ and the quantum-extremal-surface refinement [27, 28]; the HaPPY code [31] and entanglement-wedge reconstruction [32] are rigorous toy-model theorems realizing "bulk operator = equivalence class of boundary representatives"; the surface-code $H_1(\Sigma; \mathbb{F}_2)$ logical-operator theorem [36] is a rigorous instance of "cohomology protects information". The finite gluing obstruction $\check{H}^1$ of Proposition 5.3 is a real invariant.

[HEU] **layer.** Van Raamsdonk's "entanglement builds spacetime" [29] and the ER = EPR correspondence [30] are established as organizing *heuristics*, not as general theorems.

[SPEC] **layer.** That gravity *in general* — beyond AdS/CFT, in Lorentzian signature, with Einstein dynamics — emerges from the gluing of *condensed* informational structures is the central speculative hypothesis of the entire program. Per §7.4 of the program's knowledge base, this is doubly speculative: it composes Part III's [SPEC] "entanglement as condensed-sheaf compatibility" with Part VI's [SPEC] "gravity from gluing", and by Theorem 2.3 it gains no credibility from being stated in two Parts.

How it factors through the ladder (2). Goal C is the demand that the *entire* lower ladder close: the gluing/compatibility data (Rungs 2–3) must feed a causal reconstruction (Rung 4), a Malament topology (Rung 5), a conformal-plus-volume Lorentzian metric (Rung 6), and an Einstein-limit dynamics (Rung 7). The slogan "measurements glue, therefore geometry" is precisely the error of collapsing Rungs 4–7 into nothing; stated honestly, Goal C is the conjunction of the four [SPEC] reconstruction obligations of Remark 3.5, and by absorption (Theorem 2.3(iv)) it is [SPEC] as soon as any one of them is.

What a proof would require. (i) The condensed-cohomological definition of entanglement compatibility that Part III proposes but does not prove is the right invariant (Rungs 2–3); (ii) a causal / topological / conformal reconstruction discharging Rungs 4–6 (with Bisognano–Wichmann, Malament, and causal sets as the only anchors, none yet condensed or Lorentzian); (iii) a derivation that extremizing/varying the gluing data reproduces an Einstein-equation limit (Rung 7, the target of Part VII, with the Chamseddine–Connes spectral action [21], Jacobson's equation of state [26], and the Regge limit as the nearest *established* templates for "how an Einstein–Hilbert term can fall out of an algebraic/spectral/thermodynamic principle" — templates, not instances of this claim); (iv) generality beyond holographic duality. Composite status of Goal C: [SPEC].

The one honest sentence

The genuine, defensible content of Goal C today is: *there exist rigorous toy models* (HaPPY, surface codes, RT within AdS/CFT) *in which geometric/informational data are interchangeable, and there exists a well-behaved condensed homological framework in which a gluing-obstruction invariant $\check{H}^{\geq 1}$ is definable*; whether these combine into a general, Lorentzian, dynamical theory of gravity is open and, at present, [SPEC].

7 Results

We summarize the paper's deliverables, tagged by status.

1. **Weakest-link theorem ([EST]).** Theorem 2.3: $(W, \oplus, \mathsf{EST})$ is a bounded idempotent commutative monoid; composite status is the join of link statuses; SPEC is absorbing. Corollary 2.6: the assembled pipeline is [SPEC].

2. **Assembled pipeline ([HEU] as a physical skeleton; [EST] as a diagram of functors).** (1) and Figure 1: an explicit functorial pipeline $\mathrm{Cond(Set)} \to \mathrm{D(Cond(Ab))} \to \mathrm{States} \to \mathrm{Obs} \to (X, \preceq, g)$, each stage sourced to a Part, each arrow labeled.

3. **De-compressed terminal ladder ([HEU]/[SPEC] per rung).** (2) and Section 3.5: the terminal arrow is replaced by the seven-rung pregeometric ladder $\mathrm{Meas} \to \mathrm{Rel} \to \mathrm{CausalSite} \to \mathrm{Pregeometry} \to (X, \preceq, g) \to$ dynamics, each rung labeled and anchored to an established reconstruction theorem (Bisognano–Wichmann, Malament, causal sets, Connes), with the blocking work localized in Rungs 4–7 (Remark 3.5).

4. **Five established anchors positioned ([EST]).** Section 4 and Table 1: AQFT, TQFT, HoTT, DAG, ∞-topoi, each with a precise "borrows / does not get" ledger.

5. **Finite closure theorem ([EST]).** Theorem 5.8: the finite pipeline is a descent-satisfying functor $\mathsf{FinGraph}_w \to \mathsf{Met}$ with derived invariant $(\check{H}^0, \check{H}^1) = (A^{\pi_0}, A^{b_1})$ and emergent distance $d_\Gamma = d_D$, all code-backed.

6. **Three speculative goals dissected ([SPEC]).** Section 6: each of A (classical spacetime as a limit), B (locality as derived compatibility), C (gravity from gluing) is boxed, layered into [EST]/[HEU]/[SPEC], and equipped with a "what a proof would require" list.

7. **A weakest-link status census ([SPEC] headline).** Section 8.2: the composite status of the program's headline is, honestly, [SPEC].

8 Discussion

8.1 What has and has not been shown

We have *proved* ([EST]): the weakest-link theorem (Theorem 2.3) and the finite closure theorem (Theorem 5.8), including its descent lemma (Lemma 5.5), its cohomology computation (Proposition 5.3), and its spectral-distance identity (Theorem 5.7). We have *organized* ([EST]) five established anchors as structures the framework specializes. We have *not* shown, and explicitly do not claim: any recovery of a Lorentzian spacetime; any derivation of the Einstein equations; any condensed reconstruction theorem; or that gravity is entanglement. The gap between the [EST] results and the [SPEC] goals is no longer a single "final arrow": it is the lower segment of the ladder (2) — the relational→causal→topology→metric→dynamics rungs of Section 3.5 — together with the three frontiers of Section 6. The de-compression does not shrink the gap; it *localizes* it, and attaches each piece to a named established reconstruction theorem stating what a discharge would require.

8.2 The weakest-link status census

Claim	**Status**
Weakest-link theorem: $(W, \oplus)$ idempotent monoid, SPEC absorbing (Theorem 2.3)	[EST]
Faithful embedding $\mathrm{Top}_{\mathrm{cg}} \hookrightarrow \mathrm{Cond(Set)}$ (Proposition 3.2)	[EST]
D(Cond(Ab)) stable; solid/liquid subcategories (Proposition 3.3)	[EST]
Finite descent for $\mathrm{Obs} = \mathrm{Map}(-, A)$ (Lemma 5.5)	[EST]
Čech cohomology $\check{H}^0 = A^{\pi_0}$, $\check{H}^1 = A^{b_1}$ (Proposition 5.3)	[EST]
Path metric = finite Connes spectral distance (Theorem 5.7)	[EST]
Finite pipeline closure functor (Theorem 5.8)	[EST]
AQFT / TQFT / HoTT / DAG / ∞-topos anchors (Section 4)	[EST]
Ladder anchors: Bisognano–Wichmann (modular flow = boost, vacuum)	[EST]
Ladder anchors: Malament (causal order fixes topology + conformal metric)	[EST]
Ladder anchors: causal sets (order + counting → Lorentzian, controlled cases)	[EST]
The assembled pipeline is the correct skeleton for quantum gravity	[HEU]
Rungs 1–3: measurement → pairings → compatibility → relational (Section 3.5)	[HEU]
Microcausality reads as a derived compatibility condition (Goal B; Rung 2–3)	[HEU]
Rung 4: relational → causal order, background-free (beyond BW vacuum)	[SPEC]

(continued on next page)

Claim	Status
Rung 5: causal order $\to$ topology for the *extracted* order (beyond Malament)	[SPEC]
Rung 6: topology $\to$ Lorentzian metric (conformal [EST]; signature frontier L)	[SPEC]
Rung 7: metric $\to$ Einstein-limit dynamics (spectral action / Jacobson templates)	[SPEC]
Classical Lorentzian spacetime as a limit of the pipeline (Goal A; Rungs 5–7)	[SPEC]
Locality/causal order fully captured background-free (Goal B; Rung 4)	[SPEC]
Gravity from gluing of condensed informational structures (Goal C; Rungs 4–7)	[SPEC]
Composite: the program's headline quantum-gravity claim	[SPEC]

The last row is forced by Theorem 2.3(vii): the join of the column is SPEC, attained across the [SPEC] rungs of the terminal ladder (Rungs 4–7) rather than at any single "magic arrow". No quantity of [EST] rows — including the newly itemized ladder anchors, which are genuinely established reconstruction theorems — can change it, because each of those anchors licenses only its own classical, background-endowed setting and not the condensed, background-free rung the pipeline needs. That is the intended and correct outcome: the de-compression makes the blocking work explicit and localized without upgrading a single intermediate step.

8.3 Relations and non-identifications

We reiterate the non-identifications the synthesis must respect [38]:

- Condensed mathematics (external/analytic, sheaves on profinite sets) is *not* cohesive HoTT (internal/synthetic modal type theory) [34], nor topos quantum theory (presheaves on a context category of commutative subalgebras for quantum logic) [35]. All three use sheaf/topos technology to different ends; the pipeline's realization arrow is external/analytic.
- Connes' Riemannian, static spectral reconstruction [20] is *not* a Lorentzian, dynamical reconstruction (open); the synthesis carries this caveat forward from Part V without upgrade.
- The ladder's geometric anchors are theorems about geometry *already given*: Malament [22] presupposes a distinguishing spacetime, Bisognano–Wichmann [24] presupposes a Wightman vacuum on Minkowski space, and causal-set reconstruction [23]

presupposes a faithful sprinkling. They are *not* background-free, condensed reconstructions; the ladder cites them as [EST] precedents that state what Rungs 4–6 would have to prove, never as licenses that those rungs are already discharged.

- The RT formula / ER = EPR "entanglement builds geometry" [27, 30] is native to AdS/CFT and is *not* an established general mechanism; Goal C must not present it as one.

- Classical Tannakian reconstruction (proved) is *not* a condensed Tannakian reconstruction (conjectural); Part V's frontier C is inherited here unchanged.

8.4 Limitations

The limitations are structural. (i) Every computable/proved result is finite-dimensional or finite-graph; the passage to genuinely infinite condensed objects, where solidity/liquidity and resolvent-compactness do real work, is untouched by our theorems. (ii) The Lorentzian frontier is untouched by any established result we cite; our contribution there is limited to stating the gap and placing it on Rung 6 of the ladder. (iii) The pipeline is a diagram of functors whose *physical* interpretation is [SPEC] along the lower ladder segment (Rungs 4–7); the finite closure theorem vindicates the form, and only the Riemannian algebra→metric leg (Rung 6), not the physics. (iv) The composite headline claim inherits [SPEC] status from its weakest link and must not borrow credibility from the genuinely rigorous finite/homological mathematics that surrounds it — the entire point of Section 2 is to make this borrowing arithmetically impossible.

8.5 Outlook: hand-off to Part VII

Part VII [6] is the capstone: it presents the boxed hierarchy

$$\begin{gathered}\text{Category} \to \text{Condensed Sheaf} \to \text{Derived Object} \to \text{Representation} \\ \to \text{Observable} \to \text{Measurement} \to \text{Emergent Spacetime.}\end{gathered}$$

A correction this Part carries into the capstone: the last *arrow* there, Measurement → Emergent Spacetime, must *not* be read as a single [SPEC] step. It is the reconstruction ladder (2), Measurement → relational → causal → topology → metric → dynamics, whose upper rungs are [HEU] and whose lower rungs (Rungs 4–7) are [SPEC], each with a named [EST] anchor. Part VI hands Part VII four things: the *weakest-link theorem* as the formal device that keeps the hierarchy honest; the *finite closure theorem* as the maximal EST payload realized end-to-end; the *de-compressed ladder* that localizes the blocking work; and the *dissected goals* A/B/C as the concrete open problems. The most valuable next steps are not physical but mathematical, and the ladder pinpoints them: (a) prove or refute a *Riemannian* condensed reconstruction theorem (Part V frontier C), which gates Rungs 3–4 and Rung 6; and (b) prove or refute a condensed, background-free Bisognano–Wichmann / Malament analogue deriving a causal order from modular data (Rungs 4–5), which gates Goals A, B, and C without yet requiring the Lorentzian frontier.

9 Conclusion

This synthesis assembled five modular interfaces — condensed kinematics (Part I), condensed observables (Part II), condensed cohomology of information (Part III), condensed fields (Part IV), and geometry as a realization functor (Part V) — into one explicit pipeline,

$$\text{Condensed Objects} \to \text{Derived Category} \to \text{Quantum States}$$
$$\to \text{Observables} \to \text{Emergent Geometry},$$

and asked whether gravity can be understood as the gluing of informational structures rather than as a fundamental metric — while insisting that the terminal step "Observables → Emergent Geometry" is not atomic but the pregeometric reconstruction ladder (2). Our two established deliverables are a theorem and a theorem: the *weakest-link theorem*, which makes "a chain is only as established as its weakest link" a proved property of a bounded idempotent monoid; and a *finite closure theorem*, which realizes the entire pipeline on finite data as a descent-satisfying functor whose derived invariant is graph cohomology and whose emergent distance is the finite Connes spectral distance. Everything else — the recovery of classical spacetime, locality as a derived compatibility condition, gravity from gluing — is [SPEC], boxed, layered, and equipped with an explicit "what a proof would require".

The value of the synthesis is not that it solves quantum gravity; it does not. Its value is that it converts a suggestive vocabulary into a definite diagram with definite status labels, and — crucially — refuses the compression of "observables, then spacetime" into a single arrow. It isolates the exact terminal *segment* (the reconstruction ladder (2): relational → causal → topology → metric → dynamics) at which established mathematics ends and this program's physics begins, attaches each of its rungs to a named established reconstruction theorem (Bisognano–Wichmann, Malament, causal sets, Connes), and — through the status calculus — makes it arithmetically impossible to launder the speculation into rigor. "Measurements glue, therefore geometry" is, and is here shown to be, not a proof but a program with four localized, theorem-shaped obligations. The program's headline is honestly [SPEC]; its ingredients are honestly [EST]; and the discipline that keeps those two facts from being confused is itself the paper's first theorem.

10 The accompanying code

The repository directory `src/part6-condensed-quantum-gravity/` contains a self-contained Haskell package (compiling under `-Wall`) realizing the finite, decidable content of this Part:

- `Status.hs` — the warrant type $\{\mathsf{EST}, \mathsf{HEU}, \mathsf{SPEC}\}$, status composition $\oplus = \max$, the pipeline-status combinator of Definition 2.5, and exhaustive verification of the monoid laws of Theorem 2.3 (associativity, commutativity, identity, absorption, idempotence, monotonicity) by checking all cases.

- `Pipeline.hs` — a finite weighted graph as a discrete condensed object; the Čech differential δ and the ranks $\dim \check{H}^0 = |\pi_0|$, $\dim \check{H}^1 = b_1$ of Proposition 5.3; the observable sheaf and a descent (gluing) check for Lemma 5.5; and the intermediate rungs of the terminal ladder (2) exposed explicitly — `relationalStructure` (adjacency + obstruction $\check{H}^1$), `causalLayers` (a breadth-first causal order, the finite analogue of Malament's "order fixes topology"), and `openStars` (the topology as the open-star cover whose nerve recovers the graph) — so the toy walks relational → causal → topology → metric rather than jumping observables → distance; the path metric d_Γ (Floyd–Warshall) is the emergent distance of Definition 5.6.

- `Spectral.hs` — the finite Connes spectral distance d_D as the associated 1-Lipschitz linear program, and a check $d_D = d_\Gamma$ verifying Theorem 5.7 on explicit graphs.
- `Main.hs` — drives all demonstrations, including a de-compressed pass over the ladder rungs (relational → causal → topology → metric); prints the composed pipeline status (= SPEC), the derived invariant ($\dim \check{H}^0, \dim \check{H}^1$) of a sample graph, the descent check, and the agreement $d_\Gamma = d_D$; it thereby executes Theorem 5.8 end-to-end on a concrete example.

The code is a finite *shadow* of the paper's established content (Theorem 2.3, Theorem 5.8); it does not, and cannot, address the Lorentzian/dynamical frontiers, which are infinite-dimensional and Lorentzian respectively and remain [SPEC].

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