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Toward a Condensed Representation Theory of Physics: A Condensed Representation Principle

Part VII (capstone) of *Toward a Condensed Representation Theory of Physics*

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Abstract

This is the capstone of a modular seven-part program organized by one perspective: *physics is the study of realizations of condensed mathematical structures*. We do not present a theory. We present, and argue for the coherence of, a *research principle* and the program it organizes. The organizing device is a single boxed hierarchy. Its established core is a chain of five constructional arrows,

$$\text{Category} \to \text{Condensed Sheaf} \to \text{Derived Object}$$
$$\to \text{Representation} \to \text{Observable} \to \text{Measurement},$$

each of which is, *as a mathematical construction*, established ([EST]): sheaves on the pro-étale site, derived categories of condensed abelian groups, Tannakian reconstruction, and GNS/representation pairings. Measurement, however, does *not* directly produce spacetime; it produces *structured relations*, and geometry is reached only through a sequence of pregeometric layers. We therefore make the terminal step explicit, replacing the single "Measurement → Emergent Spacetime" arrow by the canonical reconstruction sub-ladder

$$\text{Measurement} \to \text{compatibility/descent} \to \text{relational structure}$$
$$\to \text{causal order} \to \text{topology} \to \text{metric/Lorentzian} \to \text{dynamics},$$

compactly

$$\text{Meas} \to \text{Rel} \to \text{CausalSite} \to \text{Pregeometry} \to (X, \preceq, g) \to \text{dynamics}.$$

We make the epistemic structure a theorem. Reusing the collaboration's status calculus — the bounded idempotent commutative monoid $(\{\mathsf{EST}, \mathsf{HEU}, \mathsf{SPEC}\}, \max, \mathsf{EST})$ — we prove a *Program Status Theorem*: the composite status of the hierarchy is [SPEC]; but the speculative burden is *not* a single magic arrow. It is localized precisely in the reconstruction sub-ladder's *Rung 4–7 segment* (relational → causal → topology → metric → dynamics), while every arrow up through relational structure is an established construction. Each speculative arrow is moreover *theorem-shaped*, anchored to a determination-direction result that makes it a target rather than a slogan: Malament's theorem (the causal order of a distinguishing spacetime determines its topology, smooth structure, and conformal metric), causal sets (order + counting → Lorentzian geometry in the continuum limit), Bisognano–Wichmann/Tomita–Takesaki (modular flow → boost/causal geometry), and Connes' spectral distance (algebra → Riemannian metric). We then state, as a disciplined research program, four open questions — (Q1) whether condensed sheaves furnish a background-independent language for quantum gravity; (Q2) whether *causal order* can be reconstructed from compatibility/modular data among condensed objects; (Q3) whether topology and a Lorentzian metric emerge from that causal order as an effective realization; (Q4) how the Einstein field equations and QFT dynamics could emerge as large-scale limits — and for each we separate what is already established (the anchors), what would count as progress, what a decisive result would require, and the honest current status. We also sharpen the program's headline objective into one theorem-shaped target: build an inverse system of finite causal diamonds (equivalently detector contexts or spin-network refinements), attach a condensed observable sheaf, extract compatibility/cohomology/modular data, and prove that the limit recovers causal order, then topology, then a Lorentzian metric in controlled cases. We supply one further [EST] deliverable: a finite, code-backed witness of the *shape* of a Q3/Q4 limit — the cycle-graph Laplacian spectrum rescales to the continuum circle Laplacian spectrum, and the path metric of a refining graph converges to the interval metric — exactly the discrete-to-continuum passage the speculative claims would need to instantiate, proved here only in the trivial finite case. We close with milestones, falsifiable sub-claims, and explicit non-identifications distinguishing this program from causal sets, cohesive homotopy type theory, topos quantum theory, loop quantum gravity, and string theory. The value of this paper is precisely its scaffolding and its honesty: rigorous mathematical ingredients, one small genuine theorem about the program's own logical shape, and a precise catalogue of what remains open.

Contents

1 Introduction

1.1 The governing perspective

This is Part VII, the capstone, of a modular seven-part program. Every Part, and this one, is organized by a single sentence.

The governing perspective of the program

Physics is the study of realizations of condensed mathematical structures.

The word *modular* is load-bearing. This is not a unified theory of anything; it is a stack of separately-stated interfaces between one specific piece of modern mathematics — condensed mathematics in the sense of Clausen and Scholze [7, 8] — and the structural vocabulary of physics. Each Part removes, or later restores, one piece of presupposed geometric background, states precisely what it establishes, and composes with its neighbours only through explicitly declared hand-offs. Part VII does not add a new interface. Its job is threefold: to state the organizing *principle* the six Parts collectively suggest; to display the whole program as one boxed hierarchy with an honest, arrow-by-arrow demarcation of what is established from what is speculative; and to convert the program's open ends into a precise, falsifiable-in-the-mathematical-sense research proposal.

We are emphatic from the first page that what follows is a *proposal*, not a result. The distinction is enforced structurally, not rhetorically, by the collaboration's status calculus [40]: every claim carries a warrant tag —

- [EST] (*established*): citable, proved mathematics, or a rigorously proved physics-adjacent theorem;
- [HEU] (*heuristic*): a physically motivated dictionary entry or analogy that is *not* itself a theorem;
- [SPEC] (*speculative*): an ontological or physical hypothesis of *this* program, with no established precedent doing the work claimed of it,

— and warrants compose *monotonically, worst-case wins*. Section 2 makes this a small theorem; the rest of the paper is disciplined by it.

1.2 The seven-part program in one page

We recall the six preceding Parts, since this capstone recapitulates them. Each replaced, demoted, or recovered one layer of geometric structure.

- **Part I** [1], *From Manifolds to Condensed Spaces*, replaced the smooth manifold by the condensed set as kinematic substrate. Its established seed is Scholze's comparison ([7], Prop. 1.7): the functor $\mathrm{Top}_{\mathrm{cg}} \hookrightarrow \mathrm{Cond(Set)}$ from compactly generated spaces to condensed sets is fully faithful, so manifolds are *faithfully embedded*, not discarded, and the surplus room in Cond(Set) is where Planck-scale or quantum-nonlocal structure might live.
- **Part II** [2], *Observables as Condensed Sheaves*, attached observables to condensed test objects, obtaining a (co)sheaf of observable algebras on the pro-étale site — a condensed refinement of the Haag–Kastler net [14] and of Costello–Gwilliam factorization algebras [16] — *before* any geometry is posited.
- **Part III** [3], *Condensed Cohomology and Quantum Information*, proposed information as the primitive object: entanglement read as a compatibility (cocycle) condition in condensed cohomology, with the surface-code $H_1(\Sigma; \mathbb{F}_2)$ logical-operator theorem and the HaPPY code [28] as its established (non-condensed) precedents.

- **Part IV** [4], *Fields Without Background Manifolds*, demoted the manifold on which a field lives, replacing $\varphi\colon M \to \mathbb{R}$ by a condensed-module-valued object $\varphi\colon C \to \mathrm{Cond}(\mathrm{Vect})$ landing in solid or liquid modules, whose established virtue is that — unlike topological vector spaces — they form a well-behaved abelian/stable category supporting derived tensor products and a six-functor formalism [8].

- **Part V** [5], *Geometry as a Realization Functor*, recovered geometry as a realization/fiber functor, aligning Gelfand duality, Tannakian reconstruction [12, 13], Connes' spectral reconstruction [20], Tomita–Takesaki/Bisognano–Wichmann [18, 19], and the cobordism hypothesis [11] into one schema, and isolated the single sharpest obstruction: *every established metric reconstruction is Riemannian, whereas spacetime is Lorentzian.*

- **Part VI** [6], *A Representation-Theoretic Foundation for Quantum Gravity*, assembled Parts I–V into one pipeline and proved two things: the weakest-link theorem (below), and a finite closure theorem exhibiting a finite weighted graph as a discrete condensed object whose emergent graph metric coincides with the finite Connes spectral distance. Both are [EST] and code-backed; the program's headline reading of the pipeline as quantum gravity is [SPEC].

1.3 What a research-proposal capstone is, and is not

The honesty contract of this capstone

This paper does **not** claim to have a theory of quantum gravity, to reconstruct Lorentzian spacetime, to derive the Einstein field equations, or to prove that spacetime is emergent. It states an organizing *principle*; it displays the program as a hierarchy whose speculative content is localized and boxed; and it enumerates open problems precisely. We prove exactly two mathematical statements — a Program Status Theorem (Theorem 3.1) about the program's own logical shape, and a finite discrete-to-continuum limit (Propositions 5.2 and 5.4) — and label *everything else*. A reader who takes away more than those [EST] statements and the honest open-problem catalogue has taken away more than we show.

A research proposal earns its keep in a specific currency: precise definitions, a clear statement of what is already known, and sharply posed open questions whose resolution would be recognizable and decisive. It does *not* earn its keep by suggestive vocabulary. The persistent temptation of a program like this one is to let a long train of genuinely rigorous mathematics (condensed sets are real; Tannakian duality is a theorem; the cobordism hypothesis is proved) silently "pay off" the speculative debt of a single physical hypothesis. The status calculus forbids this arithmetically (Section 2), and we lean on it throughout.

1.4 Contributions

1. **The Condensed Representation Principle, stated ([SPEC] as physics, precise as a proposal).** We state the organizing principle and formalize it as a claimed

factorization of physical content through the hierarchy (Section 3), whose terminal reconstruction step is displayed rung by rung, with each rung tied to the Part that develops it and each arrow carrying an explicit warrant.

2. **The Program Status Theorem ([EST]).** We prove (Theorem 3.1) that the hierarchy's composite status is [SPEC]; that the [SPEC] burden is localized not in a single magic arrow but in the reconstruction sub-ladder's *Rung 4–7 segment* (relational → causal → topology → metric → dynamics), with every arrow up through relational structure an established construction; and hence that upgrading precisely that segment to [HEU] (resp. [EST]) would make the entire composite [HEU] (resp. [EST]). Each speculative arrow of the segment is theorem-shaped, anchored to a determination-direction result (Malament; causal sets; Bisognano–Wichmann/Tomita–Takesaki; Connes' spectral distance). The proof is a finite computation in the status monoid and is code-backed.

3. **Four open research questions, dissected ([SPEC], honestly).** For each of Q1–Q4 (Section 4) we give the established anchors, a criterion for progress, the shape of a decisive result, and the honest current status, boxed.

4. **A finite [EST] witness of a Q3/Q4 limit.** We prove (Section 5) that the rescaled cycle-graph Laplacian spectrum converges to the continuum circle Laplacian spectrum and that the path metric of a refining interval graph converges to the Euclidean interval metric — the precise discrete-to-continuum passage the speculative claims would need to instantiate, established here only in the trivial finite/linear case and *explicitly not* as evidence for the physical claim.

5. **A research proposal (milestones, falsifiability, non-identifications).** Section 6 lists numbered milestones M1–M6, several falsifiable-in-the-mathematical-sense subclaims, and explicit non-identifications distinguishing this program from causal sets, cohesive homotopy type theory, topos quantum theory, loop quantum gravity, and string theory.

6. **A status census ([SPEC] headline).** Section 8.1 reproduces and expands the program's established-versus-speculative table so the composite [SPEC] status of the headline is unavoidable.

1.5 A note on originality and prior art

As recorded in the program's knowledge base, dedicated literature search found *no* existing work combining condensed mathematics with quantum gravity, holography, or algebraic QFT beyond the ordinary functional-analytic and operator-algebraic literature that predates condensed mathematics. We treat this absence as a fact to flag, not to conceal: Parts IV–VII stake out unoccupied territory, and every place the condensed machinery is asked to do physical work beyond what is proved is marked [SPEC]. We cite only real, independently verified literature and the three internal prior-work libraries of the collaboration [38, 39, 40], and we take care (Section 6.3) to distinguish this program from the related-but-distinct cohesive-HoTT [36] and topos-quantum-theory [37] programs, as well as from causal sets [33], loop quantum gravity [32], and string theory [34].

1.6 Outline

Section 2 recalls and proves the status calculus. Section 3 states the Condensed Representation Principle and displays the hierarchy, arrow by arrow — including the expanded terminal reconstruction sub-ladder — with the Program Status Theorem. Section 4 poses and dissects the four open questions. Section 5 proves the finite discrete-to-continuum witness. Section 6 gives the research proposal: milestones, falsifiable sub-claims, and non-identifications. Section 7 collects the results, Section 8 the discussion and status census, and Section 9 concludes. Section 10 documents the accompanying verified Haskell.

2 The status calculus, recalled and proved

The device that keeps this capstone honest is the status calculus, stated and proved in Part VI and recalled here so that Part VII is self-contained. It is a small theorem, but it is the load-bearing one: it is what makes "the program is a proposal" a provable statement about the program rather than a disclaimer.

Definition 2.1 (Warrant set and order)**.** Let $W = \{\mathsf{EST}, \mathsf{HEU}, \mathsf{SPEC}\}$ be the three-element *warrant set*, totally ordered by

$$\mathsf{EST} \;<\; \mathsf{HEU} \;<\; \mathsf{SPEC},$$

read "more established < more speculative". Assign ranks $r(\mathsf{EST}) = 0$, $r(\mathsf{HEU}) = 1$, $r(\mathsf{SPEC}) = 2$, so that $s \leq t \iff r(s) \leq r(t)$. Define *status composition* $\oplus \colon W \times W \to W$ by $s \oplus t = \max_<(s,t)$, the element of larger rank.

Theorem 2.2 (Weakest-link theorem [6, 40])**.** $(W, \oplus, \mathsf{EST})$ *is a commutative monoid that is idempotent and bounded, with* EST *a two-sided identity and* SPEC *absorbing, and* $\oplus$ *is monotone in each argument. Consequently, for any finite family* $s_1, \dots, s_n \in W$*,*

$$\bigoplus_{i=1}^{n} s_i \;=\; \max_{<}\{s_1, \dots, s_n\},$$

so $\bigoplus_i s_i = \mathsf{EST}$ *iff every* $s_i = \mathsf{EST}$*, and* $\bigoplus_i s_i = \mathsf{SPEC}$ *iff some* $s_i = \mathsf{SPEC}$*.*

Proof. Everything is a property of the binary maximum on a totally ordered set, transported through the rank order-isomorphism $r \colon (W,<) \xrightarrow{\sim} (\{0,1,2\},<)$, under which $r(s \oplus t) = \max(r(s), r(t))$. Associativity and commutativity are those of max; the least element is the identity ($\max(r(s),0) = r(s)$) and the greatest is absorbing ($\max(r(s),2) = 2$); idempotence is $\max(a,a) = a$; monotonicity is $a \leq a' \Rightarrow \max(a,b) \leq \max(a',b)$. Induction on n using associativity reduces $\bigoplus$ to the n-ary maximum, and $\max_i r(s_i) = 0$ (resp. $= 2$) iff all (resp. some) $r(s_i)$ vanish (resp. equal 2); reading ranks back through r^{-1} gives the warrant identities. $\square$

Definition 2.3 (Status of a pipeline)**.** A *pipeline* is a finite composable chain $A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} A_n$, each arrow f_i carrying a status $\sigma(f_i) \in W$. Its *composite status* is $\sigma(f_n \circ \cdots \circ f_1) := \bigoplus_{i=1}^{n} \sigma(f_i) = \max_{<}\{\sigma(f_1), \ldots, \sigma(f_n)\}$.

Remark 2.4 (Why a semilattice, not a group). $(W, \oplus)$ has no inverses: once a chain contains a [SPEC] link, no amount of downstream established mathematics can remove it, because SPEC is absorbing. This is exactly the epistemic content we want in a capstone, where the temptation to let rigorous ingredients launder a speculative headline is strongest. Status assignment is a monoid homomorphism from the free monoid of pipeline composites to the bounded join-semilattice $(W, \oplus, \mathsf{EST}, \mathsf{SPEC})$.

We will apply Theorem 2.2 to the expanded hierarchy in Section 3.3, where the computation localizes the program's entire speculative burden in the reconstruction sub-ladder's Rung 4–7 segment.

3 The Condensed Representation Principle

3.1 The Principle

The six Parts collectively suggest one organizing statement, which we elevate to a named principle. It is a *principle* in the sense of a proposed methodological posture and factorization claim, not a theorem; its physical content is [SPEC], and we mark it so.

The Condensed Representation Principle [SPEC]

Every physical situation is the *realization* of a condensed mathematical structure, and physical content factors through the hierarchy whose established core is

$$\underbrace{\text{Category}}_{\text{language}} \to \underbrace{\text{Condensed Sheaf}}_{\text{kinematics}} \to \underbrace{\text{Derived Object}}_{\text{invariants}} \to \underbrace{\text{Representation}}_{\text{states}}$$
$$\to \underbrace{\text{Observable}}_{\text{quantities}} \to \underbrace{\text{Measurement}}_{\text{pairings}},$$

after which *measurement does not directly produce spacetime*: it produces structured relations, and geometry is reconstructed only through the pregeometric sub-ladder

$$\text{Measurement} \;\to\; \text{compatibility/descent} \;\to\; \text{relational structure}$$
$$\to\; \text{causal order} \;\to\; \text{topology} \;\to\; \text{metric } (X, \preceq, g) \;\to\; \text{dynamics},$$

whose intermediate rungs carry, in order, gluing/descent data, relational invariants (correlations and modular flow), a causal order, a topology, a (conformal $\to$) Lorentzian metric, and finally Einstein-limit dynamics. Compactly, the program-gap is the composite arrow

$$\text{Meas} \;\to\; \text{Rel} \;\to\; \text{CausalSite}$$
$$\to\; \text{Condensed/Derived Pregeometry} \;\to\; (X, \preceq, g) \;\to\; \text{dynamics}.$$

Geometry is an *output*, obtained by realization from relational and measurement data, rather than a presupposed input. The mathematical constructions realizing the core arrows and the first steps of the sub-ladder (through relational structure) are established; the physical claim that these are the rungs of physical reality, and in particular that the reconstruction segment
relational $\to$ causal $\to$ topology $\to$ metric $\to$ dynamics can be carried out condensed-natively, is speculative.

The principle is a hypothesis about *where geometry sits in the order of explanation*. Standard physics places a manifold first and builds fields, states and observables on top of it. The principle inverts this: it places a category and a condensed sheaf first, treats states and observables as representation-theoretic data, and asks geometry to appear last, as a realization. This is exactly the inversion already achieved rigorously in four established corners — Gelfand duality, Tannakian reconstruction, Connes' reconstruction, and Bisognano–Wichmann — each of which recovers a geometric or dynamical object from purely algebraic/categorical data. The principle's speculative move is to conjecture that the *whole* of spacetime physics admits such an inversion, in a *condensed* setting, with the surplus room of Cond accommodating what manifolds cannot.

3.2 The expanded hierarchy, arrow by arrow

We now display the hierarchy with, beside each arrow, its warrant (as a mathematical construction) and the Part that develops it. Crucially, the terminal step is *not* a single arrow. Measurement produces *structured relations*, not a spacetime; geometry is reconstructed only through a sequence of pregeometric layers, which we display explicitly as a reconstruction sub-ladder. This is the single most important structural figure of the entire seven-part series.

We describe each arrow, its established anchor, and its two warrants: a *construction warrant* σ_c (the status of the arrow as a mathematical map between the indicated categories) and an *interpretation warrant* σ_i (the status of reading that map as a rung of physical reality). This two-layer bookkeeping is what keeps the demarcation honest: the core arrows A_1–A_5 and the first two reconstruction arrows R_1, R_2 are established constructions, and we say exactly where the physical reading is weaker and exactly where — the segment R_3–R_6 — no construction yet exists.

A_1**: Category $\to$ Condensed Sheaf (Parts I–II).** The passage from an ambient category of "spaces" to sheaves on the pro-étale site of a point.

A_1 anchor

The category Cond(Set) of condensed sets is (equivalently) the category of small sheaves on the pro-étale site of a point, i.e. on profinite sets with finite jointly surjective covers, computable already on extremally disconnected (Stonean) profinite sets [7]. The comparison $\mathrm{Top}_{\mathrm{cg}} \hookrightarrow \mathrm{Cond(Set)}$ is fully faithful ([7], Prop. 1.7), and Cond(Ab) is a Grothendieck abelian category, repairing the failure of topological

$$\mathrm{Cat} \xrightarrow{\substack{A_1\\ \text{[EST]}}} \mathrm{Cond(Sh)} \xrightarrow{\substack{A_2\\ \text{[EST]}}} \mathrm{D}(-) \xrightarrow{\substack{A_3\\ \text{[EST]}}} \mathrm{Rep} \xrightarrow{\substack{A_4\\ \text{[EST]}}} \mathrm{Obs} \xrightarrow{\substack{A_5\\ \text{[EST]}}} \mathrm{Meas}$$

$$\begin{tikzcd}
\text{Meas (sub-rung 1)} \arrow[d, "R_1\quad \text{[EST]/[HEU]}"] \\
\text{Compatibility/descent (2)} \arrow[d, "R_2\quad \text{[EST]/[HEU]}"] \\
\text{Relational structure (3)} \arrow[d, "R_3\quad \text{[SPEC]}"] \\
\text{Causal / order / locality (4)} \arrow[d, "R_4\quad \text{[SPEC]}"] \\
\text{Topology (5)} \arrow[d, "R_5\quad \text{[SPEC]}"] \\
\text{Metric } (X, \preceq, g) \text{ (6)} \arrow[d, "R_6\quad \text{[SPEC]}"] \\
\text{Dynamics / Einstein limit (7)}
\end{tikzcd}$$

Figure 1: The expanded hierarchy of the Condensed Representation Principle. *Top:* the established core A_1–A_5 (all [EST] as constructions). *Bottom:* the reconstruction sub-ladder R_1–R_6 replacing the old single arrow "Meas $\to (X, g)$", with local sub-rung numbers (1)–(7). Arrows R_1, R_2 are established constructions with a heuristic reading; the *Rung 4–7 segment* R_3–R_6 (relational $\to$ causal $\to$ topology $\to$ metric $\to$ dynamics) carries the whole speculative burden, each arrow anchored to a determination-direction theorem (Table 1). The full rung descriptions are given in the body text; the composite status is [SPEC] (Theorem 3.1).

abelian groups to form one. The pro-étale site is the point-specialization of the Bhatt–Scholze pro-étale topology for schemes [9], the technical ancestor engineered to repair pathologies of the étale site; sites, sheaves and descent are classical (SGA4; [10]).

Construction warrant $\sigma_c(A_1)$ = [EST]. Interpretation warrant $\sigma_i(A_1)$ = [EST]: reading a condensed sheaf as "kinematic data before geometry" is a definitional stipulation, not a further physical claim, and Part I's comparison theorem makes "manifolds are a special case" literally true.

A_2: **Condensed Sheaf → Derived Object (Parts III–IV).** Passage to the derived category of the condensed/solid/liquid abelian category, extracting cohomological invariants.

A_2 anchor

$\mathrm{D}(\mathrm{Cond}(\mathrm{Ab}))$, $\mathrm{D}(\mathbb{Z}_{\mathrm{solid}})$ and the derived categories of liquid vector spaces are stable ∞-categories with well-behaved derived tensor products and a six-functor formalism [8, 11]. Condensed/solid cohomology is a genuine derived functor; the surface-code $H_1(\Sigma; \mathbb{F}_2)$ theorem and Čech cohomology of a finite nerve ([6], closure theorem) are finite [EST] instances.

$\sigma_c(A_2)$ = [EST]; $\sigma_i(A_2)$ = [EST] as a construction of invariants, but the Part III physical reading "entanglement is a cocycle condition" is itself [SPEC] and is not part of this arrow.

A_3: **Derived Object → Representation (Part V).** Tannakian-style reconstruction of a representation category from a tensor category with a fiber functor.

A_3 anchor

For a neutral Tannakian category $\mathcal{C}$ over $\Bbbk$ with fiber functor $\omega\colon \mathcal{C} \to \mathrm{Vect}_{\Bbbk}^{fd}$, the functor $R \mapsto \mathrm{Aut}^{\otimes}(\omega_R)$ is representable by an affine group scheme G_ω and ω induces $\mathcal{C} \simeq \mathrm{Rep}_{\Bbbk}(G_\omega)$ [12, 13]. The category $\mathrm{MTM}(\Bbbk)$ of mixed Tate motives is a worked instance with Betti/de Rham/Hodge/étale fiber functors [39].

$\sigma_c(A_3)$ = [EST] *for the classical construction.* Crucial caveat: the *condensed* Tannakian reconstruction Part V actually wants — fiber functors valued in condensed/solid/liquid modules, reconstructing an emergent *geometry* rather than an affine group scheme — is *not* an existing theorem. So $\sigma_i(A_3)$ = [HEU]: the classical arrow is established, the physically relevant condensed upgrade is conjectural (this is Milestone M2, Section 6.1).

A_4: **Representation → Observable (Parts II, V).** Producing observable quantities from representation data by pairing states against operators.

A_4 anchor

The GNS construction, the spectral theorem, and the Haag–Kastler/Brunetti–Fredenhagen–Verch net formalism [14, 15] rigorously produce observable algebras and expectation values from representations and states; the representation-stack formalism [38] records that this pairing must retain automorphism (gauge) data, via $\mathrm{Aut}_{[X/G]}(x) \simeq \mathrm{Stab}_G(x)$.

$\sigma_c(A_4)$ = [EST]; $\sigma_i(A_4)$ = [HEU]: that fundamental observables are condensed-sheaf-valued *prior to* geometry (rather than manifold-local C^*-algebra-net-valued) is a physical hypothesis, Part II's central proposal, not a theorem.

A_5: Observable → Measurement (Parts II, III). Passage from an observable to the measurement pairings it records against states — expectation values and multi-point correlation functionals, the raw data from which relations (and eventually geometry) are to be reconstructed.

A_5 anchor

Given a state ω on an observable algebra, the GNS construction and the spectral theorem produce measurement statistics — expectation values $\omega(A)$, spectral projections, and multi-point correlation (Wightman) functionals $\omega(A_1 \cdots A_k)$ [14]. These pairings are the recorded output of measurement and are rigorously defined from representation/observable data.

$\sigma_c(A_5)$ = [EST]; $\sigma_i(A_5)$ = [HEU]: that these condensed measurement pairings, taken *prior to* geometry, are the complete raw material from which spacetime is to be reconstructed is a physical hypothesis, Parts II–III's proposal, not a theorem.

The reconstruction sub-ladder R_1–R_6. Measurement does not directly produce spacetime. What follows is the canonical chain of intermediate reconstruction arrows that the old single arrow "Meas → (X, g)" compressed. Its first two arrows are established constructions; its last four — the *Rung 4–7 segment* — carry the entire speculative burden, but each is *theorem-shaped*, anchored to a determination-direction result.

R_1: Measurement → Compatibility / descent data (Parts II, III). Assembling local measurement pairings into gluing data: which local outcomes agree on overlaps, and with what obstruction.

R_1 anchor

Compatibility of local sections on overlaps, and the obstruction to gluing them, are the content of descent and Čech cohomology on a site (SGA4; [10]), realized condensed-natively on the pro-étale site [7]; the surface-code $H_1(\Sigma; \mathbb{F}_2)$ theorem and the finite-nerve Čech computation of Part VI [6] are finite [EST] instances. Extracting compatibility/descent data from a family of measurement pairings is thus an established construction.

$\sigma_c(R_1) =$ [EST]; $\sigma_i(R_1) =$ [HEU]: that physical compatibility of measurements is exactly sheaf-theoretic descent on the condensed site is Part III's proposal, not a theorem.

R_2: Compatibility / descent → Relational structure (Parts III, V). Passage from raw descent data to the relational invariants a reconstruction can consume: correlations, commutation/noncommutation, context inclusions, modular flow, entanglement / relative-entropy data, and the descent obstructions themselves.

R_2 anchor

Each named datum is an established construction. Tomita–Takesaki theory canonically produces the modular automorphism group and modular flow of a von Neumann algebra with a cyclic separating vector [18]; Araki relative entropy and modular Hamiltonians are rigorously defined [14]; the context category of commutative subalgebras (context inclusion) is the Isham–Döring datum [37]; entanglement-wedge reconstruction [29] and the Ryu–Takayanagi formula [24] exhibit entanglement/relative-entropy data carrying geometric information. Producing these relational invariants from compatibility data is [EST].

$\sigma_c(R_2) =$ [EST]; $\sigma_i(R_2) =$ [HEU]: that this bundle of relational invariants, computed condensed-natively, *suffices* to determine a geometry is the hypothesis, not the theorem.

R_3: Relational structure → Causal / order / locality (Parts V, VI). Extraction of a causal partial order ("x precedes y") and a locality/commutation structure from the relational data — the first genuinely speculative reconstruction step.

R_3: relational → causal [SPEC] (anchor: modular flow → boost geometry)

The determination-direction anchor is Bisognano–Wichmann/Tomita–Takesaki: for a wedge algebra in the vacuum, the modular flow *is* the geometric boost preserving the wedge, so modular data already encodes a piece of causal geometry [19, 18]; Haag duality ties an algebra's commutant to the causal complement [14]; the ER=EPR reading connects entanglement to causal connectivity [26, 27]. But these *presuppose* the wedge geometry; recovering a causal order from relational data with *no* spacetime input has no established theorem (cf. Franco–Eckstein's algebraic causality [22]). Hence $\sigma_c(R_3) = \sigma_i(R_3) =$ [SPEC], theorem-shaped by the modular anchor.

R_4: Causal / order → Topology (Parts V, VI). Reconstruction of the point-set (and smooth) topology of the emergent space from the causal order alone.

R_4: causal → topology [SPEC] (anchor: Malament; causal sets)

The anchor makes this the sharpest leg: Malament's theorem shows that the causal (chronological) structure of a distinguishing spacetime *determines* its topology, its smooth structure, and its conformal metric [23] — causal order is almost all of geometry. Causal set theory [33] realizes "order + number = geometry" discretely, a discrete-to-continuum precedent for recovering topology from order. What is open is performing this reconstruction *from condensed relational data* rather than from an order already known to embed in a spacetime. Hence $\sigma_c(R_4) = \sigma_i(R_4) =$ [SPEC], theorem-shaped by Malament's determination.

R_5: Topology → Metric / (conformal →) Lorentzian geometry (Parts V, VI). Promotion of causal/topological data to a full metric: a conformal factor fixed by a volume (counting) datum, yielding Lorentzian $(X, \preceq, g)$.

R_5: topology → metric [SPEC] (anchor: order + volume; Connes distance)

Two determination anchors bracket this leg. Causal sets supply the conformal-to-metric completion: order fixes the conformal class (Malament) and *volume by counting* fixes the remaining scale, so order + counting yields a Lorentzian geometry in the continuum limit in controlled cases [33]. On the algebra side, Connes' spectral distance recovers a genuine metric from a spectral triple — but a *Riemannian* one [20]; the finite closure theorem of Part VI [6] instantiates the algebra → metric leg finitely. The open problem is the *Lorentzian* algebra → metric reconstruction. Hence $\sigma_c(R_5) = \sigma_i(R_5) =$ [SPEC], theorem-shaped by the causal-set and Connes anchors.

R_6: Metric → Dynamics / Einstein-limit physics (Parts IV, VI). Emergence of dynamical equations — the Einstein field equations, or QFT equations of motion — as large-scale limits of a condensed-representation dynamics on the reconstructed geometry.

R_6: metric → dynamics [SPEC] (anchor: templates, not instances)

Three determination-direction templates exist but none is condensed-native: the Chamseddine–Connes spectral action's heat-kernel expansion yields Einstein–Hilbert plus the Standard Model bosonic sector [21]; the large-spin limit of the EPRL spin-foam amplitude yields the Regge action [31, 32]; and Jacobson's $\delta Q = T\,\delta S$ together with the Faulkner et al. entanglement first law yield the (linearized) Einstein equations thermodynamically [30, 25]. Emulating any of these condensed-natively is open. Hence $\sigma_c(R_6) = \sigma_i(R_6) =$ [SPEC].

Table 1 summarizes.

Arrow	Construction	σ_c	σ_i	Part(s) / anchor
Established core				
A_1	Cat $\to$ Condensed Sheaf	[EST]	[EST]	I, II / pro-étale site
A_2	Condensed Sheaf $\to$ Derived Object	[EST]	[EST]	III, IV / D(Cond(Ab))
A_3	Derived Object $\to$ Representation	[EST]	[HEU]	V / Tannakian
A_4	Representation $\to$ Observable	[EST]	[HEU]	II, V / GNS, AQFT
A_5	Observable $\to$ Measurement	[EST]	[HEU]	II, III / GNS, correlators
Reconstruction sub-ladder (replacing the old Meas $\to (X, g)$)				
R_1	Meas $\to$ Compatibility/descent	[EST]	[HEU]	II, III / descent, Čech
R_2	Compat. $\to$ Relational structure	[EST]	[HEU]	III, V / modular flow, RT
R_3	Relational $\to$ Causal/order	[SPEC]	[SPEC]	V, VI / Bisognano–Wichmann
R_4	Causal $\to$ Topology	[SPEC]	[SPEC]	V, VI / Malament; causal sets
R_5	Topology/conformal $\to$ Lorentzian metric	[SPEC]	[SPEC]	V, VI / order+vol.; Connes
R_6	Metric $\to$ Dynamics	[SPEC]	[SPEC]	IV, VI / spectral action; EPRL

Table 1: The arrows of the expanded hierarchy, with construction warrant σ_c, interpretation warrant σ_i, and the developing Part(s) and established anchor. The core A_1–A_5 and the first two reconstruction arrows R_1, R_2 are established constructions (a merely heuristic physical reading of A_3–A_5, R_1, R_2). The whole [SPEC] burden lives in the segment R_3–R_6 (relational $\to$ causal $\to$ topology $\to$ metric $\to$ dynamics), i.e. sub-rungs 4–7, each anchored to a determination-direction theorem rather than left as a slogan.

3.3 The Program Status Theorem

The point of the two-layer bookkeeping is that it makes a precise, provable statement about the shape of the program's uncertainty.

Theorem 3.1 (Program Status Theorem [EST])**.** *Let $H = (A_1, \ldots, A_5, R_1, \ldots, R_6)$ be the expanded hierarchy of Figure 1, with construction warrants*

$$\sigma_c = (\underbrace{[\mathsf{EST}], [\mathsf{EST}], [\mathsf{EST}], [\mathsf{EST}], [\mathsf{EST}]}_{A_1–A_5},\ \underbrace{[\mathsf{EST}], [\mathsf{EST}]}_{R_1, R_2},\ \underbrace{[\mathsf{SPEC}], [\mathsf{SPEC}], [\mathsf{SPEC}], [\mathsf{SPEC}]}_{R_3–R_6})$$

and interpretation warrants

$$\sigma_i = ([\mathsf{EST}], [\mathsf{EST}], [\mathsf{HEU}], [\mathsf{HEU}], [\mathsf{HEU}],\ [\mathsf{HEU}], [\mathsf{HEU}],\ [\mathsf{SPEC}], [\mathsf{SPEC}], [\mathsf{SPEC}], [\mathsf{SPEC}])$$

in the monoid $(W, \oplus)$ *of Theorem 2.2. Write* $S = \{R_3, R_4, R_5, R_6\}$ *for the* reconstruction (Rung 4–7) segment *and* $H^- = (A_1, \ldots, A_5, R_1, R_2)$ *for the hierarchy truncated through relational structure. Then:*

(i) *the composite construction status is* $\bigoplus_k \sigma_c = \mathsf{SPEC}$*, and the composite interpretation status is* $\bigoplus_k \sigma_i = \mathsf{SPEC}$*;*

(ii) *in both layers,* SPEC *is attained exactly on the segment* S*: every arrow of* S *is* [SPEC]*, and every arrow outside* S *(all of* A_1*–*A_5, R_1, R_2*) has rank* ≤ 1*;*

(iii) *the truncated hierarchy* H^- *has composite construction status* EST *and composite interpretation status* HEU*;*

(iv) *consequently the entire speculative burden of the program is localized in the segment* S*: replacing* $\sigma(R_3) = \cdots = \sigma(R_6)$ *by* HEU *makes the composite* HEU *in both layers, and replacing all four by* EST *makes the construction-layer composite* EST*. Because each arrow of* S *carries its own determination-direction anchor (Table 1), the localization is a* sequence of four theorem-shaped targets*, not a single magic arrow.*

Proof. By Theorem 2.2 (vii) every composite is the $<$-maximum of its links; all four clauses are then finite evaluations of max, verified exhaustively in the accompanying Haskell (Section 10).

(i) Both tuples contain [SPEC] (rank 2) in the R_3–R_6 coordinates, so $\bigoplus_k \sigma_c = \bigoplus_k \sigma_i =$ [SPEC].

(ii) In each tuple the value [SPEC] occurs exactly in the four coordinates R_3, R_4, R_5, R_6; every coordinate among A_1–A_5, R_1, R_2 has rank ≤ 1. Hence the rank-maximal links are exactly the segment S in both layers.

(iii) Dropping the four S-coordinates,

$$\max([\mathsf{EST}], [\mathsf{EST}], [\mathsf{EST}], [\mathsf{EST}], [\mathsf{EST}], [\mathsf{EST}], [\mathsf{EST}]) = [\mathsf{EST}] \quad \text{(construction)},$$
$$\max([\mathsf{EST}], [\mathsf{EST}], [\mathsf{HEU}], [\mathsf{HEU}], [\mathsf{HEU}], [\mathsf{HEU}], [\mathsf{HEU}]) = [\mathsf{HEU}] \quad \text{(interpretation)},$$

which are the truncated construction and interpretation statuses of H^-.

(iv) By (ii) the segment S contains every rank-maximal link, so once A_1–A_5, R_1, R_2 are fixed at rank ≤ 1 the composite equals

$$\max\bigl([\mathsf{HEU}] \text{ or } [\mathsf{EST}],\ \max_{R \in S} \sigma(R)\bigr),$$

which is nondecreasing in $\max_{R \in S} \sigma(R)$. Setting all four S-statuses to [HEU] gives construction composite $\max([\mathsf{EST}], \ldots, [\mathsf{EST}], [\mathsf{HEU}]) = [\mathsf{HEU}]$ and interpretation composite $\max([\mathsf{HEU}], \ldots, [\mathsf{HEU}]) = [\mathsf{HEU}]$; setting all four to [EST] gives construction composite [EST]. □

Corollary 3.2 (The program is a proposal, provably)**.** *The Condensed Representation Principle, as a physical claim about spacetime, has status* [SPEC]*, and does so because of the reconstruction segment* R_3*–*R_6*. No accumulation of established mathematics in* A_1*–*A_5, R_1, R_2 *can change this, because* [SPEC] *is absorbing (Remark 2.4). Equivalently: the program is exactly* four theorem-shaped reconstruction steps — *relational* $\to$ *causal* $\to$ *topology* $\to$ *metric*

$\to$ dynamics — away from being at worst heuristic rather than speculative; and by Malament's theorem the first of these (recovering causal order) already almost determines the next two.

Remark 3.3 (This is the value proposition of the whole series). Theorem 3.1 is deliberately modest: it proves nothing about physics. What it proves is that the seven-part program has a *clean logical shape* — seven established constructional arrows and a sharply identified speculative *segment* of four determination-direction reconstruction steps — rather than a diffuse cloud of hand-waving. A research program's first obligation is to know precisely where its hardest unknowns live and in what order they must fall. For this program the answer is the segment R_3–R_6: reconstruct, in turn, causal order (from relational/modular data), then topology, then a Lorentzian metric, then dynamics. Malament's theorem collapses the middle of this segment — once causal order is recovered, topology and the conformal metric follow — so the true first target is R_3 (relational $\to$ causal). Everything in Section 4 is a facet of this segment.

4 Four open research questions

We now state the program as four open questions. Each is a facet of the reconstruction segment R_3–R_6 (relational $\to$ causal $\to$ topology $\to$ metric $\to$ dynamics) and of the condensed upgrades of the earlier arrows. For each we give: the established anchors, a criterion for *progress*, the shape of a *decisive* result, and the honest current status. We state the questions precisely enough that a positive or negative answer would be recognizable, and we tie each to the specific reconstruction arrow it targets.

4.1 Q1: A background-independent language for quantum gravity

Question 4.1 (Background independence)**.**
[SPEC] Do condensed sheaves furnish a *background-independent* language for quantum gravity, in the sense that the observable content of a quantum field theory can be specified as a (co)sheaf on the pro-étale site of a point, with no spacetime manifold among the inputs, so that a spacetime, where one exists, is recovered only as an *output* rather than assumed as data?

What is already established for Q1

Locally covariant QFT (Brunetti–Fredenhagen–Verch [15]) is already a functor $\mathcal{A}\colon \mathsf{Loc} \to C^*\text{-Alg}$ from globally hyperbolic spacetimes to C^*-algebras that is "background-independent" in the weak sense of being defined over *all* spacetimes at once. The comparison $\mathrm{Top}_{\mathrm{cg}} \hookrightarrow \mathrm{Cond(Set)}$ [7] and the well-behaved abelian category Cond(Ab) provide the target site; the Gwilliam–Rejzner comparison [17] relates nets and factorization algebras for free fields. These are [EST].

Progress would be a condensed refinement $\mathcal{A}^{\mathrm{Cond}}$ of the BFV functor whose *source* is a category of condensed test objects (not Loc), together with a proof that $\mathcal{A}^{\mathrm{Cond}}$ restricted to the image of Loc recovers the ordinary net — i.e. that the condensed formalism *specializes* to Haag–Kastler on globally hyperbolic spacetimes. *A decisive result* would be a condensed-sheaf-of-observables formalism with no spacetime input that reproduces a non-trivial (ideally interacting) QFT and from which the underlying spacetime is an output. *Honest status:* [SPEC]. The substrate (condensed sheaves) is [EST]; that it *suffices* to encode QFT background-independently is unproved. This is the condensed upgrade of arrow A_4 and Milestone M1.

4.2 Q2: Causal structure from compatibility conditions

Question 4.2 (Causal reconstruction, arrow R_3)**.**
[SPEC] Can the *causal order* "x precedes y" be reconstructed from the relational data of Figure 1 — correlations, commutation/noncommutation, context inclusion, and above all *modular flow* and relative-entropy data — among condensed observable-sheaves, so that locality and causal order emerge as a *derived* relation (R_3) rather than a presupposed one? By Malament's theorem [23] this single relation then almost determines topology (R_4) and the conformal metric (R_5).

What is already established for Q2

Bisognano–Wichmann recovers Lorentz boosts as the modular flow of a wedge algebra [19, 18]; Haag duality relates an algebra's commutant to the causal complement [14]. Malament's theorem [23] shows that the causal (chronological) structure of a spacetime determines its conformal metric and topology — a rigorous statement that *causal order is nearly all of geometry*. Causal set theory [33] realizes "order + number = geometry" discretely. Franco–Eckstein [22] give an algebraic formulation of causality in noncommutative geometry. All are [EST].

Progress would be a definition of a relational/compatibility functor on condensed observable-sheaves together with a theorem that, in a controlled model (e.g. the finite/discrete condensed objects of Part VI, or a free field on a fixed globally hyperbolic spacetime), the induced partial order coincides with the known causal order. *A decisive result* would be a condensed causal-order reconstruction theorem valid with *no* spacetime input, from which — by Malament's theorem — topology and the conformal metric follow, going beyond the wedge-presupposing Bisognano–Wichmann [19] and complementing the *Riemannian* Connes reconstruction [20]. *Honest status:* [SPEC], with an unusually sharp target thanks to Malament's theorem: one does not need a full metric first, only the causal order. This is arrow R_3, the head of the segment R_3–R_6, and Milestone M4.

4.3 Q3: Classical spacetime as an effective realization

Question 4.3 (Effective realization, arrows R_4–R_5)**.**
[SPEC] Do the *topology* (R_4) and a *Lorentzian metric* (R_5) of classical spacetime arise as an effective realization of the reconstructed causal order — order fixing the conformal class and a counting (volume) datum fixing the scale — so that a smooth (pseudo-)Riemannian or

Lorentzian manifold arises as a controlled scaling limit of a family of condensed/derived objects, the manifold being an approximation valid at large scales with corrections computable at small scales?

What is already established for Q3

Manifolds embed faithfully: $\mathrm{Top}_{\mathrm{cg}} \hookrightarrow \mathrm{Cond}(\mathrm{Set})$ [7]. The finite closure theorem of Part VI [6] produces an emergent graph metric equal to the finite Connes spectral distance. Gromov–Hausdorff convergence of refining graphs to manifolds, and the convergence of graph Laplacian spectra to manifold Laplacian spectra, are classical analysis (we prove the circle case in Section 5). These are [EST].

Progress would be a controlled limit theorem in which a family of *condensed* objects C_n (not merely finite graphs) produces, in a scaling limit, a smooth manifold with metric, with an error estimate — a condensed analogue of Gromov–Hausdorff or spectral convergence. *A decisive result* would be an effective-field-theory-style expansion in which classical spacetime geometry is the leading term of a systematic expansion of a condensed-representation-theoretic object, with the first correction computed and, ideally, of a form comparable to a known quantum-gravity correction. *Honest status:* [HEU]/[SPEC]. The embedding and the finite/spectral limits are [EST]; "effective realization" as a *controlled physical* limit is [HEU] at best and the headline physical claim is [SPEC]. These are arrows R_4–R_5 and Milestone M5. Section 5 supplies the finite, honest witness of the *shape* of such a limit and nothing more.

4.4 Q4: Familiar equations as large-scale limits

Question 4.4 (Emergent dynamics, arrow R_6)**.**
[SPEC] How could familiar dynamical equations — the Einstein field equations, or QFT equations of motion — emerge as *large-scale limits* of a condensed-representation dynamics on the reconstructed geometry (R_6): is there an action functional or variational principle on condensed-representation data whose stationarity, in a large-scale limit, yields these equations, with the Newton constant and couplings computed rather than inserted?

Two established templates for Q4

Two existing results are the nearest *templates* — not instances — for what Q4 asks. (a) The Chamseddine–Connes *spectral action* $\mathrm{tr}\, f(D/\Lambda)$ has a heat-kernel expansion whose leading terms reproduce the Einstein–Hilbert action plus the Standard Model bosonic sector [21]; this is [EST] as a computation, [HEU] as physics. (b) The large-spin (semiclassical) limit of the EPRL spin-foam vertex amplitude reproduces the Regge action, hence discretized general relativity [31, 32]; [EST]/[HEU]. A third, thermodynamic template: Jacobson [30] derives the Einstein equations from $\delta Q = T\,\delta S$ across local Rindler horizons, and Faulkner et al. [25] derive *linearized* Einstein equations from the entanglement first law in holography; [HEU]/[SPEC] as general mechanisms.

Progress would be a condensed-representation action whose stationarity reproduces even a *free*-field equation of motion condensed-natively (no manifold), or a linearized wave equation, as a large-scale limit. *A decisive result* would be a derivation of the linearized (and then full) Einstein field equations as the large-scale limit of a condensed-representation dynamics, with the emergent G_N computed. *Honest status:* [SPEC]. We present the spectral action and the spin-foam and thermodynamic derivations as templates the program would need to *emulate condensed-natively*, explicitly not as achievements of the program itself. Per the status-composition discipline (Theorem 3.1, Remark 2.4), composing Part IV's (mathematically [EST], physically [SPEC]) condensed QFT with this [SPEC] dynamical claim yields [SPEC]. This is arrow R_6 and Milestone M6.

4.5 The four questions are facets of the reconstruction segment

Proposition 4.5 (Localization of the open problem [EST])**.** *Each of Q1–Q4 is a strengthening of either the condensed interpretation of a core/early arrow (in $\{A_3, A_4, A_5, R_1, R_2\}$) or of an arrow of the reconstruction segment $S = \{R_3, R_4, R_5, R_6\}$; and each has status* [SPEC] *whose sole rank-maximal contributions lie in S. Consequently, by Theorem 3.1 (iv), a positive resolution of the segment S — of which Q2 supplies the head arrow R_3 and Q3, Q4 the arrows R_4–R_6 — would, together with the established A_1–A_5, R_1, R_2, lift the whole program's construction-layer status from* [SPEC] *to at most* [HEU]*.*

Proof. Q1 strengthens the condensed reading of A_4/R_1 (observables and descent without geometry); Q2 is the relational $\to$ causal content of R_3; Q3 is the causal $\to$ topology $\to$ metric content of R_4, R_5; Q4 is the dynamics content of R_6. In each case the rank-maximal ([SPEC]) links lie in the segment S. By Theorem 3.1 (ii) and (iv), S is exactly the set of rank-maximal links of the whole hierarchy; downgrading all of S to [HEU] makes the construction-layer composite max([EST], . . . , [EST], [HEU]) = [HEU]. ☐

This proposition is the precise sense in which the program is *one hard segment*, not four unrelated problems. It is also why Q2 — causal reconstruction (R_3), sharpened by Malament's theorem to "reconstruct the causal *order*, not the full metric" — is the natural first target: recovering R_3 collapses R_4 and the conformal part of R_5 by Malament.

5 A finite [EST] witness: discrete to continuum

Questions Q3 and Q4 both hinge on a discrete-or-condensed structure reproducing a continuum quantity in a scaling limit. We prove here the smallest honest instance of that passage: a family of finite graphs (discrete condensed objects) whose spectral and metric invariants converge to those of a continuum. We emphasize immediately what this does and does not show.

What this section is, and is not

The results below are elementary and fully [EST]. They are *not* evidence for the physical claims of Q3/Q4. They establish only that the *shape* of a discrete-to-continuum limit — discrete Laplacian spectrum → continuum Laplacian spectrum; discrete path metric → continuum metric — is realized in the trivial linear/finite case. A physically relevant Q3/Q4 limit would require condensed (not merely finite) objects, a nonlinear (dynamical) content, and a Lorentzian target. We prove none of that here.

5.1 The cycle graph as a discrete condensed circle

Let C_n be the cycle graph on vertices $\mathbb{Z}/n$ with unit edge weights, regarded as a finite (hence discrete) condensed object with vertices placed at $j/n \in \mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$. Its graph Laplacian Δ_n acts by $(\Delta_n f)_j = 2f_j - f_{j-1} - f_{j+1}$.

Lemma 5.1 (Cycle Laplacian spectrum [EST]). *The eigenvalues of Δ_n are $\lambda_{n,k} = 2 - 2\cos(2\pi k/n)$ for $k = 0, 1, \dots, n-1$, with eigenvectors $v_j^{(k)} = e^{2\pi ikj/n}$.*

Proof. $(\Delta_n v^{(k)})_j = (2 - e^{-2\pi ik/n} - e^{2\pi ik/n})v_j^{(k)} = (2 - 2\cos(2\pi k/n))v_j^{(k)}$, using $e^{i\theta} + e^{-i\theta} = 2\cos\theta$. The $v^{(k)}$ are the characters of $\mathbb{Z}/n$ and form an orthogonal basis, so these are all n eigenvalues. □

The continuum comparison object is the circle $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ with Laplacian $-\mathrm{d}^2/\mathrm{d}x^2$, whose eigenvalues are $\mu_k = (2\pi k)^2$ with eigenfunctions $e^{2\pi ikx}$.

Proposition 5.2 (Spectral convergence [EST]). *Fix $k \in \mathbb{Z}$. Rescale the discrete Laplacian by the squared inverse mesh $h^{-2} = n^2$. Then*

$$\Lambda_{n,k} \;:=\; n^2\,\lambda_{n,k} \;=\; n^2\big(2 - 2\cos(2\pi k/n)\big) \;\xrightarrow[n\to\infty]{}\; (2\pi k)^2 \;=\; \mu_k,$$

with the explicit error bound $0 \le \mu_k - \Lambda_{n,k} \le \dfrac{(2\pi k)^4}{12\,n^2}$ for $n \ge 2|k|$.

Proof. Write $\theta = 2\pi k/n$. Consider the even function $g(\theta) = 2 - 2\cos\theta$, which has the convergent alternating series expansion

$$g(\theta) = \theta^2 - \frac{\theta^4}{12} + \frac{\theta^6}{360} - \cdots = \sum_{m\ge1}(-1)^{m-1}\frac{2\,\theta^{2m}}{(2m)!}.$$

The ratio of consecutive terms is $\theta^2/\big((2m+1)(2m+2)\big) \le \theta^2/12$ for all $m \ge 1$, so the terms are strictly decreasing in magnitude precisely when $\theta^2 < 12$, i.e. $|\theta| < \sqrt{12} \approx 3.46$. In that range the standard alternating-series bracketing (a partial sum ending in a subtracted term underestimates, one ending in an added term overestimates) gives $\theta^2 - \frac{\theta^4}{12} \le g(\theta) \le \theta^2$. Multiplying by n^2 and substituting $\theta^2 = (2\pi k)^2/n^2$ and $\theta^4 = (2\pi k)^4/n^4$ gives

$$(2\pi k)^2 - \frac{(2\pi k)^4}{12n^2} \;\le\; \Lambda_{n,k} \;\le\; (2\pi k)^2.$$

The hypothesis $n \ge 2|k|$ ensures $|\theta| = 2\pi|k|/n \le \pi < \sqrt{12}$, safely within that radius, so the bracket applies; the squeeze then gives both convergence and the stated bound. □

Remark 5.3 (Reading of Proposition 5.2). The discrete object C_n knows nothing of the circle; yet its rescaled spectrum reproduces the circle's to relative error $O(k^2/n^2)$. This is exactly the *form* an "effective realization" (Q3) would take — a manifold invariant recovered as a scaling limit of a discrete/condensed invariant — and exactly the form the "emergence of dynamics" (Q4) would need for a Laplacian-type operator. It is also exactly as far as elementary mathematics takes us: the target here is Euclidean and static, not Lorentzian and dynamical.

5.2 The refining interval and its metric limit

Let P_n be the path graph on vertices $\{0, 1, \dots, n\}$ with each edge of weight $1/n$, regarded as a discrete condensed object with vertex j placed at $j/n \in [0, 1]$. Its path metric is $d_n(i, j) = |i - j|/n$.

Proposition 5.4 (Metric (Gromov–Hausdorff) convergence [EST])**.** *For all $x, y \in [0, 1]$, writing $i = \lfloor nx \rfloor$, $j = \lfloor ny \rfloor$,*

$$\big|\, d_n(i, j) - |x - y| \,\big| \;\le\; \frac{2}{n} \;\xrightarrow[n\to\infty]{}\; 0.$$

Consequently the finite metric spaces (P_n, d_n) converge in the Gromov–Hausdorff sense to the interval $([0, 1], |\cdot|)$, and the emergent path metric of the discrete condensed object recovers the continuum metric.

Proof. $d_n(i, j) = |i - j|/n$ and $|x - y|$ differ by at most $\big|\,|i - j| - n|x - y|\,\big|/n$. Since $|i - nx| < 1$ and $|j - ny| < 1$, we get $\big|\,|i - j| - |nx - ny|\,\big| < 2$, hence the bound $2/n$. The vertex set $\{j/n\}$ is a $\frac{1}{n}$-net of $[0, 1]$, so the correspondence $j \leftrightarrow j/n$ has Gromov–Hausdorff distortion $\le 2/n \to 0$. $\square$

Together, Propositions 5.2 and 5.4 give a finite, code-backed (Section 10) witness that discrete condensed objects can carry, in a scaling limit, both a continuum spectrum and a continuum metric — the two data a realization functor would have to produce. They are the honest floor under Q3/Q4, and nothing above it.

6 The research proposal

We now cast the program as a proposal: numbered milestones, falsifiable sub-claims, and explicit non-identifications. This is what distinguishes a program from a slogan.

The sharp research target: from slogan to theorem-shaped program [SPEC]

The program's headline objective is not "spacetime is emergent" — a slogan — but a single theorem-shaped target that instantiates the reconstruction segment R_3–R_5 in a controlled model:

> *Build an inverse system of finite causal diamonds (equivalently, detector contexts or spin-network refinements); attach a condensed observable sheaf*

to it; extract its compatibility / cohomology / modular data (R_1, R_2); and prove that the limit recovers the causal order (R_3), then the topology (R_4), then a Lorentzian metric (R_5) in controlled cases.

This target is strictly smaller than "derive general relativity" and strictly larger than any finite check we prove here: it is exactly the segment R_3–R_5 made into a limit theorem over an explicit inverse system, with Malament's theorem [23] doing the causal $\to$ topology $\to$ conformal work once R_3 is in hand and a counting (volume) datum fixing the Lorentzian scale in the manner of causal sets [33]. The milestones M1–M6 below decompose it; the finite witness of Section 5 is its trivial static shadow.

6.1 Milestones

Each milestone is a self-contained, recognizable mathematical target. We give, for each, its status and which arrow/question it addresses.

Milestone 6.1 (Condensed net comparison, Q1/A_4, R_1)**.** Define a condensed refinement $\mathcal{A}^{\mathrm{Cond}}$ of the Brunetti–Fredenhagen–Verch functor whose source is a category of condensed test objects, and prove that its restriction to (the image of) Loc recovers the Haag–Kastler net on globally hyperbolic spacetimes. *Target status if achieved:* [EST] comparison theorem; would upgrade the *interpretation* of A_4 from [HEU] toward [EST]. *Current:* open.

Milestone 6.2 (Condensed Tannakian reconstruction, Q1/A_3)**.** Construct a neutral *condensed* Tannakian category — a rigid tensor category with a fiber functor valued in condensed/solid/liquid modules — and prove a reconstruction theorem recovering an affine group scheme (*no geometry yet*) analogous to the classical case [12]. *Target status:* [EST]. *Current:* open; no condensed fiber functor is constructed in the literature.

Milestone 6.3 (Inverse system recovers causal order, Q2/R_1–R_3 — *the first target*)**.** Build an inverse system of finite causal diamonds (equivalently detector contexts or spin-network refinements), with transition maps the standard AQFT inclusions of local regions ordered by containment (a directed set under refinement), attach a condensed observable sheaf, and extract its compatibility / Čech-cohomology / modular data (R_1, R_2). Prove that in a controlled model (finite condensed objects, or a free field on a fixed globally hyperbolic spacetime) the induced relation recovers *exactly the known causal order* (R_3), with modular flow supplying the determination-direction anchor (Bisognano–Wichmann [19]). *Target status:* [EST] in the model; [HEU] as a general mechanism. *Current:* open. This is the smallest theorem-shaped instance of the sharp target.

Milestone 6.4 (Condensed Lorentzian reconstruction, Q2–Q3/R_3–R_5 — *the pivot*)**.** Prove a reconstruction theorem that carries the segment R_3–R_5: from purely condensed relational / spectral data, recover the causal *order* (R_3), then — by Malament's theorem [23] — the topology (R_4) and the conformal metric, then fix the Lorentzian scale by a counting (volume) datum in the manner of causal sets [33], yielding $(X, \preceq, g)$ (R_5). This extends Connes' *Riemannian* theorem [20] to Lorentzian signature and goes beyond the wedge-presupposing Bisognano–Wichmann [19]. *Target status:* [EST] would instantiate the segment R_3–R_5 at

[EST] and, with M1–M3, lift the program's construction-layer composite from [SPEC] to [EST] (Theorem 3.1 (iv)). *Current:* open; this is the program's central hard problem.

Milestone 6.5 (Condensed scaling limit to a manifold, Q3/R_4, R_5)**.** Prove a controlled limit theorem in which a family of condensed objects converges (in a Gromov–Hausdorff or spectral sense) to a smooth (pseudo-)Riemannian or Lorentzian manifold with its topology and metric (R_4, R_5), generalizing Propositions 5.2 and 5.4 beyond finite graphs to the inverse system of M3. *Target status:* [EST] (Riemannian); the Lorentzian and physical versions remain [SPEC]. *Current:* open beyond the finite case proved here.

Milestone 6.6 (Emergent linearized Einstein equations, Q4/R_6)**.** Exhibit a condensed-representation action functional whose stationarity, in a large-scale limit, yields the linearized Einstein field equations (and, as a first step, a free wave equation), with the emergent Newton constant computed, emulating condensed-natively the spectral-action [21] and spin-foam [31] templates. *Target status:* [SPEC] until achieved. *Current:* open.

The dependency structure is linear and honest: M1–M2 sharpen the early arrows; M3 is the first target (it instantiates R_3, causal order, on an explicit inverse system of causal diamonds); M4 is the pivot (it carries the segment R_3–R_5, the full Lorentzian reconstruction); and M5, M6 build the effective-limit and dynamical content (R_4–R_6) on top of M4. Proposition 4.5 says the segment $S = \{R_3, \dots, R_6\}$ is the crux, and Malament's theorem makes R_3 (=M3) its natural head.

6.2 Falsifiable sub-claims

A proposal must be refutable. We isolate sub-claims each of which is a precise mathematical statement that could be *disproved*, and whose disproof would eliminate a specific piece of the program. (We use "falsifiable" in the mathematical sense: a conjecture with a definite truth value, whose negation is a recognizable counterexample or no-go theorem.)

F1. *Specialization.* The condensed net $\mathcal{A}^{\mathrm{Cond}}$ of M1, if constructed, recovers the Haag–Kastler net on Minkowski space. *Refutation:* a proof that no condensed refinement can reproduce microcausality on Minkowski, or an explicit obstruction. This would falsify the claim that condensed sheaves are an *adequate* language for QFT (Q1).

F2. *Causal-order recovery (R_3).* The relational/modular functor of M3, applied to the inverse system of causal diamonds (or a free scalar field on 2-dimensional Minkowski space), induces the correct causal order. *Refutation:* a computation showing the induced relation is not the causal order (e.g. it is symmetric, or fails Malament's determination). This would falsify the causal-reconstruction arrow R_3 (Q2) at its simplest test.

F3. *Riemannian obstruction persists (R_5).* There exists a condensed spectral datum whose *only* metric realizations are Riemannian, with a provable obstruction to any Lorentzian realization. *If true*, F3 would be a no-go theorem showing the topology $\to$ metric arrow R_5 cannot be instantiated by the spectral-triple route, redirecting the program away from M4's spectral formulation toward the causal-set order+volume route.

F4. *No large-scale action.* Every condensed-representation action functional in a specified class has a large-scale limit that is *not* of Einstein–Hilbert form. *If true* for a natural class, F4 would falsify the dynamical claim (Q4) for that class and force a different variational principle.

None of F1–F4 is settled; each is stated so that either a positive construction (M1–M6) or a no-go result would be a definite outcome. A program whose sub-claims are all unfalsifiable is not a research program; these are.

6.3 Relations and explicit non-identifications

We position the program among established alternatives, and state precisely where it differs, so that it is not mistaken for — or dismissed as a re-labeling of — any of them. The discipline is the collaboration's non-identification rule: shared vocabulary is not shared content.

Causal set theory [33]. *Shared:* the conviction that causal order is close to all of geometry (Malament), and that a discrete substrate may underlie spacetime. *Different:* a causal set is a locally finite poset with *no* algebraic/representation-theoretic superstructure; this program's substrate is a condensed sheaf of *observables* on the pro-étale site, and causal order (if it emerges) is a *derived* compatibility relation among sheaves, not a postulated primitive. *Non-identification:* the condensed site is not a causal set, and "order" here is an output of the reconstruction arrow R_3, not an input.

Cohesive homotopy type theory [36], built on univalent foundations [35]. *Shared:* both use topos/sheaf technology and modalities to organize geometry synthetically or sheaf-theoretically. *Different:* cohesive HoTT is an *internal, synthetic* language (shape/flat/sharp modalities in a cohesive ∞-topos of smooth ∞-groupoids); condensed mathematics is an *external, analytic* construction (sheaves on extremally disconnected profinite sets), engineered for topological/analytic algebra to be abelian. *Non-identification:* the condensed ∞-topos is not a cohesive ∞-topos; "shape" in the two theories is a different functor. We do not claim a translation between them.

Topos quantum theory (Isham–Döring) [37]. *Shared:* presheaf/topos technology applied to quantum theory. *Different:* Isham–Döring use presheaves on the context category of commutative subalgebras to reformulate quantum *logic* and the Kochen–Specker theorem; this program uses the condensed/pro-étale site to reformulate *spacetime and fields*, not primarily logic. *Non-identification:* the spectral presheaf of topos quantum theory is not a condensed sheaf of observables; the two address different questions.

Loop quantum gravity [32]. *Shared:* background independence as a nonnegotiable, and the appearance of discrete geometric spectra. *Different:* LQG quantizes the gravitational connection on a fixed differentiable manifold via holonomy–flux algebras and spin networks; this program does *not* presuppose a manifold at all, placing a condensed sheaf first. *Shared template, not identity:* the EPRL large-spin limit [31] is cited (Q4) as a *template* for how a discrete formalism's semiclassical limit can reproduce the Regge action, *not* as a construction this program has carried out. *Non-identification:* a spin network is a graph labeled by SU(2) representations on a fixed manifold; a condensed sheaf on the pro-étale site is neither.

String theory [34]. *Shared:* the ambition that spacetime and its dynamics are emergent, and the use of the Ryu–Takayanagi formula and AdS/CFT [24, 27] as evidence that geometry can be reconstructed from lower-dimensional data. *Different:* string theory posits fundamental extended objects propagating in (a background or emergent) target space with a specific worldsheet dynamics; this program posits no strings and no fixed dynamics, only the condensed-representation hierarchy. *We borrow AdS/CFT's holographic reconstruction as an established* [HEU] *precedent for the segment R_2–R_5 (entanglement/relational data to geometry), not as a component. Non-identification:* the condensed sheaf of observables is not a boundary CFT, and the emergent $(X, \preceq, g)$ of R_5 is not, as constructed here, an AdS bulk.

7 Results

We collect what this capstone actually establishes, keeping the [EST]/[HEU]/[SPEC] discipline explicit.

1. **The Condensed Representation Principle and its hierarchy ([SPEC] as physics, precise as a proposal).** Sections 3.1 and 3.2 state the organizing principle and present the expanded hierarchy (Figure 1, Table 1), whose established core A_1–A_5 is extended by the reconstruction sub-ladder R_1–R_6, with each rung tied to Parts I–VI and each arrow carrying a construction warrant σ_c and an interpretation warrant σ_i. The core A_1–A_5 and the first two reconstruction arrows R_1, R_2 are established constructions; the segment R_3–R_6 (relational $\to$ causal $\to$ topology $\to$ metric $\to$ dynamics) is speculative.

2. **The Program Status Theorem ([EST]).** Theorem 3.1 proves that the composite status is [SPEC] in both warrant layers, attained exactly on the reconstruction segment R_3–R_6, so that the whole speculative burden is localized in that Rung 4–7 segment of four theorem-shaped targets rather than a single magic arrow; Corollary 3.2 concludes that the program is provably a proposal, and Proposition 4.5 that Q1–Q4 are facets of that segment.

3. **Four dissected open questions ([SPEC], honestly).** Sections 4.1 to 4.4 give, for each of Q1–Q4, the established anchors, a progress criterion, the shape of a decisive result, and the honest status. Q2 (causal reconstruction), sharpened by Malament's theorem, is identified as the natural first target.

4. **A finite discrete-to-continuum witness ([EST]).** Proposition 5.2 shows the rescaled cycle-graph Laplacian spectrum converges to the circle Laplacian spectrum with error $O(k^4/n^2)$; Proposition 5.4 shows the refining path metric converges in Gromov–Hausdorff distance to the interval metric. Both are code-backed and are the honest floor under Q3/Q4.

5. **A research proposal (mixed status).** Section 6.1 gives milestones M1–M6 with M4 (condensed Lorentzian reconstruction) as the pivot; Section 6.2 gives falsifiable sub-claims F1–F4; Section 6.3 gives explicit non-identifications from five neighbouring programs.

Nothing in this list asserts a physical result. The two genuine theorems (Theorem 3.1 and the finite limits) are about, respectively, the program's logical shape and a trivial continuum limit; the physical content is, throughout, a boxed and localized [SPEC].

8 Discussion

8.1 Status census

We reproduce and expand the program's established-versus-speculative census, one row per Part, so that the composite [SPEC] status of the headline is unavoidable to the reader. This is the epistemic-transparency device the three seed projects also use.

Part	Established core	Speculative claim	Arrow(s)	Composite
I	Condensed sets, pro-étale site, $\mathrm{Top}_{\mathrm{cg}} \hookrightarrow \mathrm{Cond}$	Planck-scale spacetime is condensed; nonlocality via profinite indexing	A_1	[HEU]/[SPEC]
II	Sites, sheaves, descent; AQFT nets; factorization algebras	Observables condensed-sheaf-valued before geometry	A_1, A_4, A_5, R_1	[HEU]/[SPEC]
III	Solid/liquid modules, derived categories, RT, HaPPY	Entanglement = sheaf compatibility; geometry from compatibility	A_2, R_1, R_2	[SPEC]
IV	Solid/liquid modules, analytic rings, six-functor formalism, vN algebras	Condensed QFT models real interacting fields	A_2, R_6	[SPEC] (math [EST])
V	Tannakian, Connes, Bisognano–Wichmann, cobordism hypothesis	Condensed-Tannakian reconstruction of emergent geometry	A_3, R_3–R_5	[SPEC]
VI	All of the above, recomposed; closure theorem	Gravity from gluing; classical spacetime as a limit	A_1–A_5, R_1–R_6	[SPEC]
VII	The full hierarchy, arrow by arrow; Program Status Theorem	Einstein equations / QFT dynamics as large-scale limits	R_6	[SPEC]

Table 2: Established mathematical core versus speculative physical claim, by Part, with the hierarchy arrow(s) each Part develops and the composite status. Every mathematical core listed is [EST]; every headline physical claim composes down to [SPEC] by the weakest-link theorem. Part VII's own contribution is to prove (Theorem 3.1) that the [SPEC] is localized in the reconstruction segment R_3–R_6.

Corollary 8.1 (Census composite [EST])**.** *Reading down the "Composite" column of Table 2 and joining in $(W, \oplus)$ yields [SPEC]. The program's mathematical ingredients are [EST]; its headline physical claims are [SPEC]. There is no row and no composition in which the headline becomes [HEU] or [EST] without first resolving the reconstruction segment R_3–R_6.*

Proof. Each "Composite" entry is $\geq$ [HEU] and at least one (rows III–VII) is [SPEC]; by Theorem 2.2 (vii) the join is [SPEC]. That no headline reaches [HEU] without the segment R_3–R_6 is Theorem 3.1 (iv) applied per row. □

8.2 Limitations

We state the limitations plainly; they are the content of the proposal, not caveats to it.

- *No physical theorem is proved.* The two [EST] results concern the program's logical shape and a trivial Euclidean limit. Neither touches Lorentzian signature, dynamics, or interacting fields.

- *The reconstruction segment R_3–R_6 has no condensed-native precedent.* Each leg is anchored to a determination-direction theorem — Bisognano–Wichmann (relational $\to$ causal), Malament (causal $\to$ topology/conformal), causal sets and Connes (topology $\to$ metric), the spectral action and spin foams (metric $\to$ dynamics) — but every such precedent is Riemannian (Connes), wedge-presupposing (Bisognano–Wichmann), continuum-embedding (Malament), or non-condensed (causal sets); the condensed Lorentzian reconstruction the segment demands is open (cf. Franco–Eckstein and successors).

- *The condensed upgrades of A_3–A_5, R_1, R_2 are unbuilt.* No condensed fiber functor, no condensed Tannakian category, and no condensed BFV net exist in the literature; M1–M3 are prerequisites, not results.

- *The dynamical templates are borrowed, not achieved.* The spectral action and spin-foam and thermodynamic derivations (Q4) are other programs' results, cited as templates the present program would need to emulate condensed-natively.

- *Composition caps everything.* By the weakest-link theorem, any claim depending on the whole pipeline is [SPEC]; no amount of rigorous condensed mathematics upstream can change this while any arrow of the segment R_3–R_6 stands.

8.3 Why a proposal, and why now

Two facts make this a reasonable time to state the proposal precisely rather than wait. First, the mathematical substrate has matured: condensed mathematics, its six-functor formalism, solid and liquid modules, and condensed homotopy theory are now developed enough to *ask* the physical questions in a well-typed way, even if not to answer them. Second, the reconstruction paradigm is now a genuine pattern across four established corners (Gelfand, Tannaka, Connes, Bisognano–Wichmann), so "geometry as an output of representation data" is no longer a metaphor but a repeated theorem in the Riemannian/static setting. The proposal is precisely the conjecture that this pattern extends, condensed-natively, to the Lorentzian/dynamical setting — and the honest observation that it has not yet been shown to.

9 Conclusion

This capstone stated the Condensed Representation Principle — *physics is the study of realizations of condensed mathematical structures* — and displayed the seven-part program as one boxed hierarchy whose established core, Category $\to$ Condensed Sheaf $\to$ Derived Object $\to$ Representation $\to$ Observable $\to$ Measurement, is extended by the pre-geometric reconstruction sub-ladder Measurement $\to$ compatibility/descent $\to$ relational $\to$ causal $\to$ topology $\to$ metric $\to$ dynamics, with each arrow's warrant and developing Part made explicit. Making the terminal step explicit is the point: measurement does not directly produce spacetime, and displaying the intermediate rungs prevents the single old arrow from hiding a chain of distinct reconstruction problems. We proved the one thing a capstone can honestly prove about a program of this kind: not that its physics is right, but that its uncertainty has a clean shape. By the Program Status Theorem, seven arrows (the core A_1–A_5 and R_1, R_2) are established constructions and the entire speculative burden is localized in the Rung 4–7 segment R_3–R_6: reconstruct causal order (from relational/modular data), then topology, then a Lorentzian metric, then dynamics. The four open questions are facets of that segment; the milestones make M3 (an inverse system of causal diamonds recovering causal order) the first target and M4 (the full Lorentzian reconstruction R_3–R_5) the pivot; Malament's theorem sharpens the segment to the recovery of a causal *order* rather than a full metric; and the finite discrete-to-continuum witness shows, in the only case elementary mathematics settles, the exact shape a realization limit would take.

The program is, by the discipline it imposes on itself, a proposal and not a theory. Its value is the scaffolding and the honesty: rigorous ingredients, one small true theorem about its own logic, four sharply posed questions, six milestones, four falsifiable sub-claims, and five explicit non-identifications. Whether the reconstruction segment can be instantiated — whether an inverse system of condensed causal diamonds recovers causal order, and then a Lorentzian metric in controlled cases — is left open, deliberately and precisely. That is the point of a capstone made of gaps: to say exactly which gaps matter, and in what order they must fall.

10 The accompanying verified Haskell

The repository accompanying this paper contains a small Haskell package, located in the directory

`src/part7-condensed-representation-principle/`,

that mechanically checks the finite [EST] content of this capstone. It compiles clean under `ghc -Wall` and consists of four modules.

- `Status` realizes the warrant monoid $(W, \oplus, \mathsf{EST})$ of Theorem 2.2 and verifies its laws (associativity, commutativity, identity EST, absorbing SPEC, idempotence, monotonicity, worst-case) *exhaustively* over the finite carrier, so the verification is a decision procedure, not a spot check.
- `Hierarchy` encodes the expanded hierarchy of Figure 1 — the established core A_1–A_5 and the reconstruction sub-ladder R_1–R_6 — as a typed pipeline, each arrow carrying its name, developing Part, construction warrant σ_c and interpretation warrant σ_i. It computes the composite construction and interpretation statuses, identifies the blocking segment $\{R_3, R_4, R_5, R_6\}$ as the exact set of rank-maximal arrows, and computes

the truncated statuses — realizing every clause of the Program Status Theorem (Theorem 3.1) as an executable check.

- `Limit` realizes the discrete-to-continuum witness of Section 5: it computes the cycle-graph Laplacian eigenvalues (Lemma 5.1), their rescaling $\Lambda_{n,k}$, the continuum eigenvalues $\mu_k = (2\pi k)^2$, and the errors, verifying the monotone bound of Proposition 5.2; and it computes the path-metric error of Proposition 5.4, checking it is $\leq 2/n$.

- `Main` runs both demonstrations: it prints the hierarchy with per-arrow statuses, the composite/blocking/truncated results of the Program Status Theorem, and the spectral- and metric-convergence tables showing the errors shrinking as n grows.

The code is intended as a machine-checked companion to the two [EST] theorems of this paper. It makes no physical claim; like the paper, it computes only the finite, established content and prints an explicit reminder that the emergent quantities are Euclidean/finite, not Lorentzian spacetime.

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